Topological defects in condensed matter and cosmology

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Abstract

Topological defects are artifacts of phase transition and arise from the topology of the order parameter space. They are ubiquitous in physics. Because they are generic, seemingly different physical systems, but having the same symmetrical and topological properties, exhibit the same topological defects. In this paper the connection between phase transition, symmetry and order will be explored. Topological defects will be defined and characterized using rudimentary homotopy group theory. Common examples of topological defects will be studied from condensed matter and cosmology. No prior knowledge of topology or topological defects from the readers is assumed.

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I. OVERVIEW

Phase transitions are often associated with a spontaneous symmetry breaking, a process by which a lower symmetry, ordered, ground state appears out of a higher symmetry, disordered state. If the characteristic correlation length of the system is small compared to the system size, then spontaneous symmetry breaking will inevitably lead to the creation of certain types of defects defined as regions of high symmetry phase (false vacuum) surrounded by an ordered phase (true vacuum). These are called topological defects because their existence is intimately connected to the topology of the order parameter space that corresponds to a degenerate vacuum. By studying the topology of the order parameter space one can characterize and predict these defects.

In this paper I will discuss the physics of topological defects by looking at a variety of system of different dimensionalities. The paper will be organized as follows: First, the phenomenon of spontaneous symmetry breaking associated with phase transition will be discussed. Then the concept of order parameter will be introduced. The topology of the order parameter space will be discussed with the help of various illustrative examples. The mathematics of topological defects will be discussed and the concepts of homeomorphism and homotopy will be introduced to the reader. The formation of topological defects and its connection to the topology of the order parameter space will then be discussed for simple illustrative systems. The Kibble mechanism of topological defect formation will be introduced. Examples of these defects are ubiquitous in physics; they can be found in condensed matter physics and cosmology alike. Examples from this broad spectrum of physical systems will be treated all the while keeping the general mathematical theory of topological defects in sight.

II. PHASE TRANSITION AND SPONTANEOUS SYMMETRY BREAKING

When a system, in the absence of an external ordering field, assumes a state that has a lower symmetry than the original system, it is said to have experienced a spontaneous symmetry breaking. A common example is the liquid to solid phase transition in water. Let us write the liquid-phase Hamiltonian, $\mathscr{H}_{\text{liquid}}$, in a rather general way:

$$\mathscr{H}_{\text{liquid}} = T + \frac{1}{2}\sum_{i \neq j} V_{\text{liquid}}(r_{ij}), \tag{1}$$

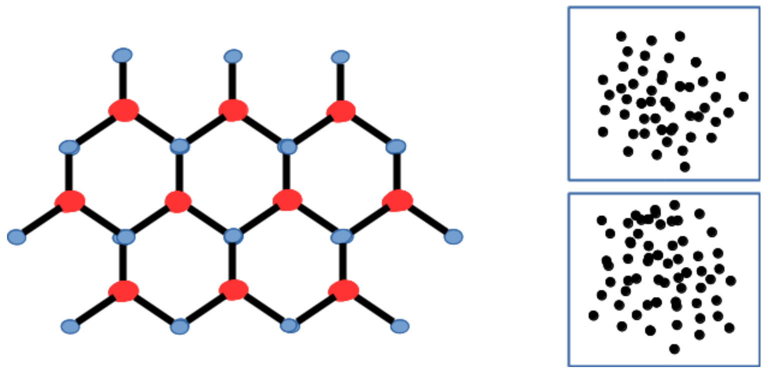

FIG. 1. One of many stable structures of ice (left) [1]. The crystalline structure is invariant under discrete translations of integer times the lattice constants and also under fixed angle rotations. The distributions of water molecules at two different locations (right) look the same. Liquid water exhibits continuous translational and rotational invariance.

where T is the kinetic energy containing no space dependence, i, j label water molecules and $V(r_{ij})$ is some two-body interaction potential energy that depends of the seperation r_{ij} of two water molecules. The potential term contains the information about the spatial symmetry of the phase.

Water looks the same everywhere - it has continuous translational symmetry. It also looks the same when one rotates it through an arbitrary angle - it has continuous rotational symmetry. Mathematically, the action of continuous translation and rotation operators will leave $V_{\text{liquid}}(r_{ij})$, and also $\mathscr{H}_{\text{liquid}}$, invariant. Ice, however, is a crystalline structure - it breaks continuous translational and rotational symmetry. A comparison of the structure of ice and liquid water is given in Fig. 1.

In the ice phase the potential energy takes a special form which encodes the crystalline structure of ice: $V_{\text{liquid}}(r_{ij}) \to V_{\text{ice}}(r_{ij})$ which is now invariant under discrete translation and rotation operations, but not under continuous ones. When the crystallization of ice begins, water molecules start aligning themselves in space choosing the bonding directions *arbitrarily*, making the symmetry breaking *spontaneous*.

Note that in the discussion above, $V_{\text{ice}}(r_{ij})$ is a special case of the more general $V_{\text{liquid}}(r_{ij})$, and we can say that ice is a lower symmetry, ordered phase of a higher symmetry, disordered liquid phase Hamiltonian.

Let us look at another example - the 2d Ising model on a square lattice laid out on the

$x - y$ plane. For this system the Hamiltonian in the field-free case is

$$\mathscr{H}_{\text{Ising}} = -J \sum_{<i,j>} \sigma_i^z \sigma_j^z, \tag{2}$$

where the spin coupling $J > 0$, i, j label a lattice site, $\sigma_i^z = \pm 1$ and $<>$ denotes restricted sum over nearest neighbors only.

This Hamiltonian is a simple, yet rich, description of ferromagnetic materials. We notice that under the global spin-flip transformation $\sigma_i^z \rightarrow -\sigma_i^z, \forall i$, $\mathscr{H}_{\text{Ising}}$ remains invariant. In the high temperature limit, the spins on the lattice are randomly distributed and the ensemble average spin per site, $\bar{\sigma}$, vanishes. This highly disordered state is also highly symmetric because in this phase $\bar{\sigma}$ is invariant under the spin-flip transformation - the full symmetry of the Hamiltonian is retained in the ground state. As this system is cooled below some critical temperature, the spins on the lattice align themselves spontaneously either all facing $+z$ or all facing $-z$. In this phase, however, $\bar{\sigma} \rightarrow -\bar{\sigma}$ under the spin-flip transformation. Also note that for an infinite system, to achieve $\bar{\sigma} \rightarrow -\bar{\sigma}$, an infinite amount of energy will be required. We can see that in the ordered, ferromagnetic phase of the 2d Ising lattice the ground state spontanously breaks the full symmetry of the Hamiltonian.

III. ORDER PARAMETER SPACE

The term *order* has been used without definition in the previous section for the two cases considered. Let us now be more precise with what it means.

Consider the difference between a crystal on a monatomic square lattice and a liquid in 2d. In this crystal atoms wiggle about the lattice points. One can define a displacement $\mathbf{d}(\mathbf{r})$ as a vector that brings a wayward atom at $\mathbf{r}$ back to a lattice point. But because the lattice has discrete translational symmetry, the choice of $\mathbf{d}(\mathbf{r})$ is not unique [1]. In fact,

$$\mathbf{d}(\mathbf{r}) \equiv \mathbf{d}(\mathbf{r}) + ma\hat{x} + na\hat{y}, \tag{3}$$

where a is the lattice constant and m, n are integers.

When this crystal is melted, the lattice is destroyed, and there are no preferred positions for the atoms to fall to. Now $\mathbf{d}(\mathbf{r})$ can take any continuous value on an infinite two dimensional space. In this example, $\mathbf{d}(\mathbf{r})$ is thus an *order parameter* that differentiates the ordered crystalline phase from the disordered liquid phase.

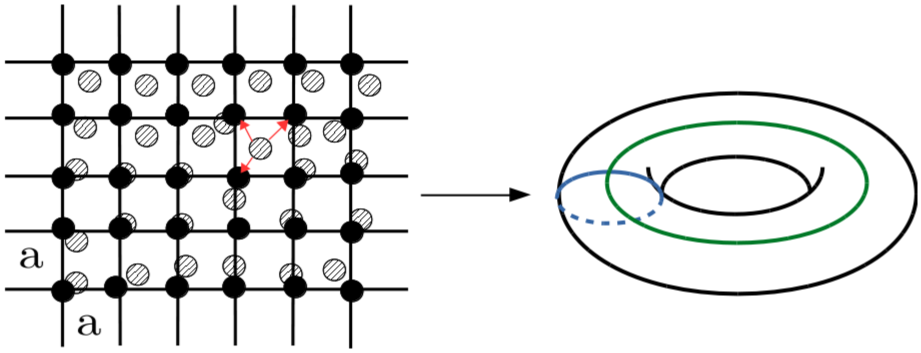

FIG. 2. The order parameter for a 2d square lattice crystal structure (left) is the displacement (shown in red arrows) that brings an off-site atom to a lattice site. Owing to the discrete translational symmetry of the underlying lattice, this order parameter has the form shown in Eq. 3. The space of this order parameter has the topology of a 2-torus, T^2, (right). The green and blue lines show a single winding around and through the central hole respectively.

In light of Eq. 3, one can see that the space of the order parameter in the crystalline phase is a 2-torus, T^2, that is, $\mathbf{d}(\mathbf{r})$ lives on the surface of a doughnut. This is shown in Fig. 2. The integers m, n tell us how many times one has to go around and through the hole, respectively, to come back to the same spot. The characteristic of the 2-torus is the single hole it contains and any smooth deformation of this torus will preserve this hole.

Revisiting the 2d Ising model, we identify the order parameter as the magnetization, $\bar{\sigma}$. In the symmetric phase, $\bar{\sigma} = 0$. But in the ordered, ferromagnetic phase, $\bar{\sigma} = \pm 1$. The space of the order parameter in the ferromagnetic phase thus consists of just two values as shown in Fig. 3.

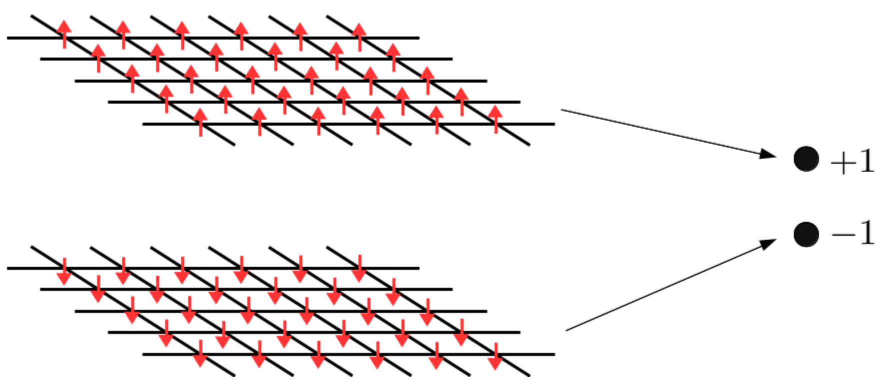

FIG. 3. Mapping of the degenerate ferromagnetic ground states (left) to the order parameter space (right) for the 2d Ising model. Red arrows indicate spin direction.

As a last example, consider a 3d ferromagnet. In this case the spins at the lattice sites can point in any direction in space. Again, in the symmetric phase the order parameter,

$\bar{\sigma} = 0$. But in the ferromagnetic phase spontaneous symmetry breaking leads to an arbitrary selection of a magnetization direction in 3d. As a result the order parameter space is a unit 2-sphere, S^2, as shown in Fig. 4.

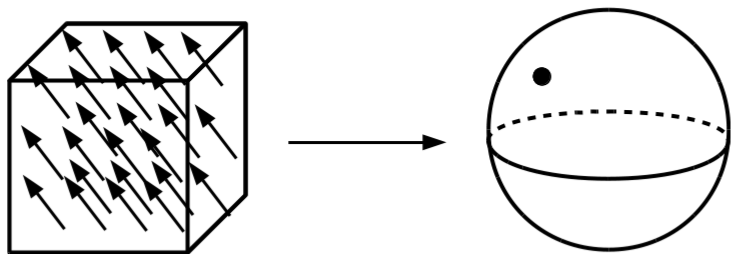

FIG. 4. Mapping of a 3d ferromagnetic ground state (left) to the order parameter space (right). The order parameter space has the topology of a 2-sphere, S^2, since the magnetization direction is arbitrarily chosen during spontaneous symmetry breaking. The particular direction of magnetization on the left gets mapped onto a point (shown as a black dot) on the surface of the sphere on the right.

IV. TOPOLOGICAL DEFECTS

So far we have seen examples of spontaneous symmetry breaking associated with phase transition and that for different systems the order parameter space has different topology in the ordered state. In this section the concept of topological defects will be introduced. We will define what a topological defect is, and how we can characterize them. We start with the mathematics of it -

A. Homotopy group

Topology is a classification of space. Roughly speaking, two spaces have the same topology if they can be continuously deformed into one another. The following joke gives an example: *In Cafe Topology, when you ask for a doughnut you may be served a cup of coffee.* A coffee cup and a doughnut are topologically the same (or *homeomorphic*) and their topology is characterized by the single hole they both contain in them. Similarly, a circle and a triangle are topologically equivalent.

The topology of the underlying space dictates what types of loops can be drawn on it [2]. This is illustrated in Fig. 5 where we consider two 2d topological spaces - X, which has no

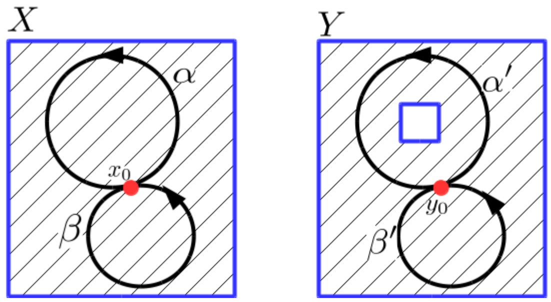

FIG. 5. X and Y, being topologically distinct, allow different classes of loops to live on them. On X, loops α and β can be shrunk to the point x_0 that joins them - α and β belong in the same class of loops. On Y, loops α' and β' cannot be shrunk to the point y_0 that joins them because of the presence of the hole (small blue square) - α' and β' belong to different classes of loops.

holes, and Y, which has a single hole in it. On X, the loops labelled α, β joined at point x_0 can be shrunk to a point. This criterion is used to classify the loops, and one says that α and β are *homotopic*. In contrast, on Y, the loops α', β' joined at point y_0 cannot both be shrunk to a point because α' goes around the hole. Thus, α' and β' are non-homotopic. The dependence on the fixed point x_0 can be removed in a connected space as one can draw a line between any two fixed points and then shrink to a single point [3]. Because an infinite number of such loops can be drawn (with different numbers of winding, directionality of the looping, at different locations, of different shapes, etc.), one defines a homotopy class of loops instead of dealing with individual loops. All sets of homotopy classes on a single topological space form a group denoted by π_N, where N is the degree of the homotopy group given by the dimensionality of the loops drawn on the space ($N = 1$ for the loops in Fig. 5, $N = 2$ for a closed surface that may be used to wrap a 2-sphere, etc.). This can be seen by first defining the class of all loops that shrink to a point to be the identity element, $\mathbb{I}$. Product of two classes of loops give another class of loops. And classes of oppositely moving loops can be multiplied to give the identity. A 2-sphere has a trivial first homotopy group, denoted by $\pi_1(S^2) = \mathbb{I}$, because any 1d loop on the surface of a 2-sphere can be shrunk to a point. However, the second homotopy group of S^2 is not trivial as one cannot shrink the wrapping paper to a point after completely wrapping a football. As another example, consider the topology of a circle, S^1. One can make an integer number of loops on the circle, and so the first homotopy group is non-trivial and is given by $\pi_1(S^1) = \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers. The non-triviality of the homotopy group is indicative of topological defects, as will be demonstrated in the rest of this section.

With this (pedestrian) introduction to topological spaces and homotopy group out of the way, we are ready to look at topological defects. For our first few encounters with topological defects, we will find the defect and then make connections to homotopy theory. However, we will eventually turn the problem on its head and begin to predict types of topological defects by applying of homotopy theory first and then look for the solution.

B. Domain wall

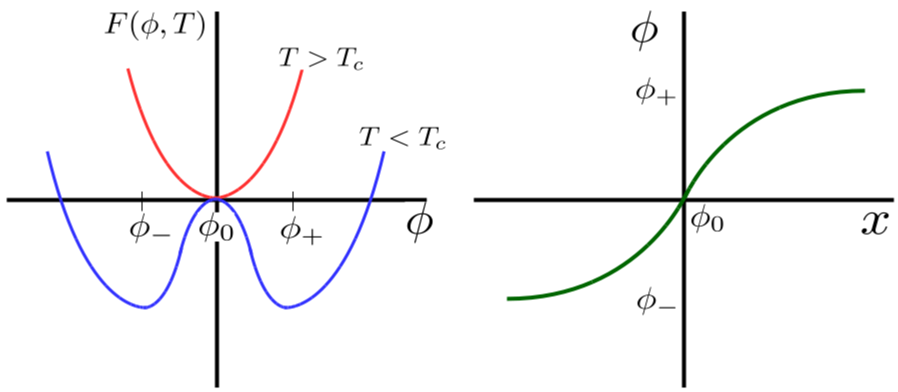

FIG. 6. On the left, free energy curves show development of the degenerate vacuum states as the temperature is decreased. On the right is a one dimensional representation of a wall defect: the order parameter between differently oriented ferromagnetic domains (*true vacua*) must correspond to a region of the high symmetry phase (*false vacuum*).

Let us study again the 2d Ising model on the square lattice but this time in the phenomenological Landau-Ginzburg formalism [4]. Let us redefine the order parameter as a smooth function, ϕ. The free energy of the system in the Landau-Ginzburg theory is written as

$$F(\phi,t) = \frac{1}{2}a_1(t)\phi^2 + \frac{1}{4}a_2(t)\phi^4, \tag{4}$$

where $t \equiv T/T_c - 1$ is the dimensionless temperature scaled with respect to the disordered to ordered phase transition critical temperature, T_c and a_1, a_2 are expansion coefficients.

At the ground state we require $\partial_\phi F = 0$, and $\partial^2_\phi F > 0$. We find that for $a_1(t), a_2(t) > 0$, there is a single ground state at $\phi = 0$. However, for $a_1(t), -a_2(t) < 0$, the ground state is degenerate: $\phi^2 = -a_1(t)/a_2(t)$. The coefficients are expanded in t: $a_1(t) = b_1 t$, $a_2(t) = b_2$, where b_1, b_2 are positive. Finally, the degenerate ground state solutions become $\phi_\pm =$

$\pm\left(\frac{-b_1 t}{b_2}\right)^{1/2}$ for $t < 0$. Choosing $b_1 = b_2$ and taking the limit $T << T_c$, we reproduce the result quoted in Section II: $\phi_{\pm} = \pm 1$.

In Fig. 6 the free energies are shown for $T > T_c$ and $T < T_c$. If we assume that the temperature is lowered from above T_c so slowly that the whole system remains in thermal equilibrium at all times (the adiabatic approximation), then the whole system will either choose ϕ_+ or ϕ_- below T_c. However, if the temperature change is not infinitesimally gradual and if speed of information in the medium is finite, then regions far away from each other will choose their ground state in an uncorrelated manner. This argument that causally disconnected regions in space will choose their ground state randomly was put forward by Kibble [5]. As a result, via the so called *Kibble mechanism*, domains of ϕ_+ and ϕ_- will form separated by walls of size $\approx \zeta$, where ζ is the correlation length of the system. Between a ϕ_+ and a ϕ_- domain there must necessarily exist a ϕ_0 region. This ϕ_0 region of high symmetry, high energy phase trapped between differently ordered phases is known as a wall defect. As shown in Fig. 3, the two differently ordered domains get mapped onto two distinct points on the order parameter space that cannot be smoothly deformed into one another. The domain wall is called a topological defect because it essentially arises from the topology of the order parameter space. In this case the zeroth homotopy group of the order parameter space is non-trivial: $\pi_0 = \mathbb{Z}_2$.

C. Dislocation

The dislocation defect is an area of a crystal where one or more extra arrays of atoms enter breaking the local discrete translational invariance symmetry. One cannot get rid of a dislocation by rearranging the positions of the atoms in the neighborhood of the defect. In Fig. 7 a dislocation defect is shown on a 2d square lattice with a monatomic basis. One can detect the presence of the defect by traversing a loop on the lattice and counting the number of rows and columns of the atoms encountered. If an extra column of atoms has been inserted, and the loop encloses the defect, then one finds a mismatch in the number of columns encountered going up and down the horizontal segments of the loop. In Fig. 2, the order parameter space for the 2d square lattice has been shown. In Fig. 7 the trajectory of the order parameter on the torus is drawn as one traverses the loop on the crystal. Because the loop in real space encloses a defect, the trajectory of the order parameter is a loop around the hole on T^2. One cannot shrink this loop to a point because of the presence of the hole. If the loop in real space had not contained the dislocation defect, the order parameter trajectory would have been a loop that could be shrunk to a point. This is the reason why a dislocation is a topological defect. In general, there could be dislocations due extra rows and columns, and the number of extra rows and columns would correspond to the number of windings through and around the hole of T^2, respectively. Now, because T^2 is a connected space, its zeroth homotopy group is trivial. But, as discussed above, 1d loops may have non-trivial winding around the hole. Therefore, the first homotopy group of T^2 is non-trivial and is characterized by two integers: the number of windings around and through the hole, respectively. Thus, for T^2, $\pi_1 = \mathbb{Z} \times \mathbb{Z}$.

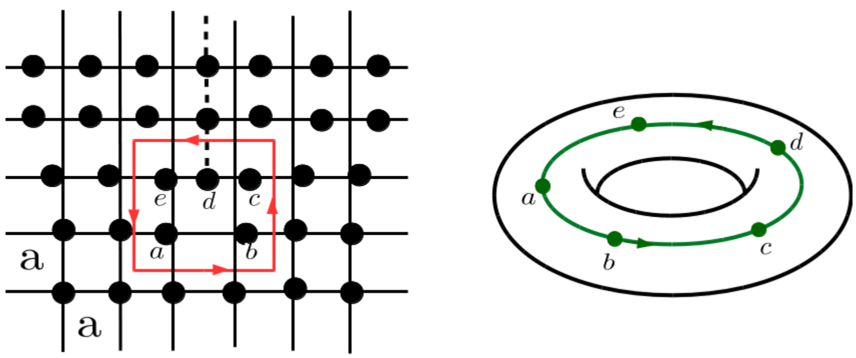

FIG. 7. On the left, an extra column of atoms (dashed line) has been inserted causing a dislocation defect. As the defected region is traversed along the red line, there is a mismatch of one atom between the $a \to b$ and the $c \to e$ segments. A full circuit along the red line, therefore, corresponds to one winding around the hole of the order parameter space shown on the right.

We are beginning to see a pattern here. Originally, we were identifying a defect before looking for its topological origin. But equipped with homotopy theory, we can now directly predict the existence of different topological defects simply by analyzing the topology of the order parameter space.

D. Hedgehog, or monopole

The mathematics of topological defects is generic, so we should be able to find them in different areas of physics. In that spirit, we switch our attention from condensed matter to cosmology and study the hedgehog, or monopole defect.

In search of the hedgehog, we need to look for an order parameter space with the topology of a 2-sphere, S^2, where the zeroth and the first homotopy groups are trivial, but the second homotopy group is non-trivial. One can have integer numbers of wrappings on the S^2, that is, $\pi_2(S^2) = \mathbb{Z}$. The existence of this non-trivial second homotopy group points to the presence of a hedgehog [6].

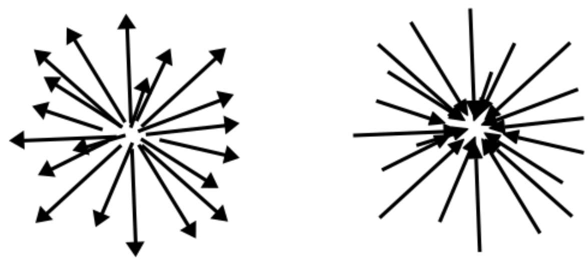

FIG. 8. Hedgehog (monopole) defect in real space corresponding to $\pi_2(S^2) = 1$ (left) and $\pi_2(S^2) = -1$ (right). Arrows indicate the Higgs field.

In one particular quantum field theory where the Higgs field transforms as a triplet of real numbers, spontaneous breaking $SO(3)$ gauge symmetry leads to the degenerate ground state manifold having the topology of S^2 [7, 8]. In this case the Hedgehog solution was found by Polyakov [9] and t'Hooft [10] independently.

The Hamiltonian of this problem is given by [9]

$$\mathscr{H} = \int \mathrm{d}^3x \left\{ \frac{1}{2}\pi_a^2 + \frac{1}{2}\left(\nabla\phi_a\right)^2 - \frac{\mu^2}{2}\phi_a^2 + \frac{\lambda}{4}\phi_a^4 \right\}, \tag{5}$$

where, ϕ_a is a Higgs triplet with $a = 1,2,3$ representing Cartesian directions, and π_a is the conjugate momentum field.

The Hamiltonian contains only even powers of ϕ_a. In other words, it depends only on the length of ϕ_a. Now, ϕ_a, being a triplet of reals, can be rotated using 3×3 rotation matrices belonging to the group $SO(3)$. But such transformations will leave the Hamiltonian invariant since rotations leave the length of a vector intact. Therefore, the Hamiltonian exhibits $SO(3)$ symmetry.

From the potential part of the Hamiltonian, $V \equiv -\frac{\mu^2}{2}\phi_a^2 + \frac{\lambda}{4}\phi_a^4$, one can draw connection to the Landau-Ginzburg free energy expression, Eq. 4, discussed earlier. In analogy, one can see that here too exists a degenerate ordered ground state which will break the full symmetry of the Hamiltonian. During this spontaneous symmetry breaking the non-zero value of the gauge is chosen to be the "$\hat{z}$-direction" of the gauge space. With this choice, the gauge field is now invariant under $SO(2)$ rotations about the reference axis - a lower symmetry than that of the Hamiltonian. The degenerate manifold of ϕ_a thus becomes the surface of a 2-sphere [7, 11]. At this point it should be emphasized that this 2-sphere is not on real space, but on the internal space of the Higgs field triplet.

This Hamiltonian admits a solution of the form [9]

$$\phi_a = x_a \frac{u(r)}{r}, \tag{6}$$

where r is the radial distance.

The equation obeyed by $u(r)$ is given in [9] and is not reproduced here. However, the important feature of $u(r)$ is that it is a constant far from the origin. As a result, in Eq. 6 ϕ_a points radially outward in real space like the spikes of a hedgehog.

Taking the Higgs field as our order parameter, covering a hedgehog in real space corresponds to non-trivially wrapping the order parameter space, S^2. In Fig. 8 the real space hedgehog configurations for $\pi_2(S^2) = \pm 1$ are shown.

E. String, or vortex

Following the trend above, we can predict that in cases where the order parameter space, or true vacuum manifold, has the topology of a one dimensional loop, S^1, the first homotopy group is non-trivial: $\pi_1(S^1) = \mathbb{Z}$. This characterizes the string or vortex defect. The vortex defect has been observed in superfluid He^4 [12]. Its cosmic counterpart - the cosmic string - has yet to be observed. We discuss both here to enjoy the underlying mathematical connection that exists between two seemingly unrelated systems.

Superfluid He^4 is described by the Bose-Einstein condensate wavefunction that takes the form $\Psi = |\Psi| e^{i\theta}$, where $e^{i\theta}$ is a $U(1)$ phase with θ being the phase angle. The potential energy for this system is given by [13]

$$V = \alpha \Psi^2 + \frac{\beta}{2} \Psi^4, \tag{7}$$

where a_1, a_3 are constants and in general has the shape of a Mexican Hat shown in Fig. 9.

This system is invariant under $U(1)$ transformations of the Ψ since only even powers enter the potential. The system obeys the Gross-Pitaevski equation [13]

$$i\hbar \partial_t \Psi = -\frac{\hbar^2}{2m} \nabla^2 \Psi - \left(\frac{\partial V}{\partial |\Psi|^2} \right) \Psi, \tag{8}$$

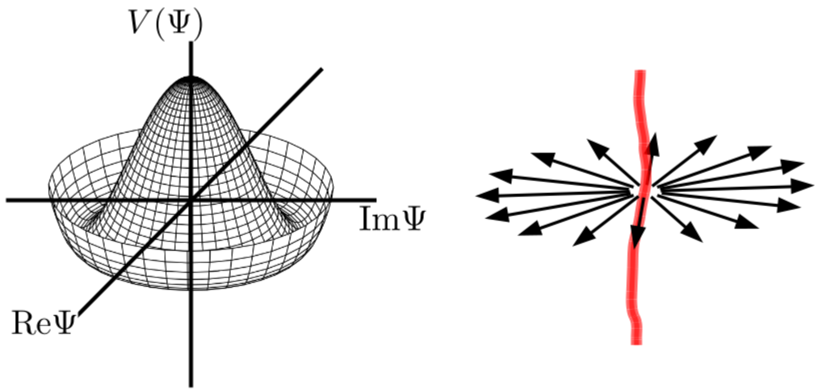

FIG. 9. The Mexican hat potential [14] (left) showing S^1 degenerate vacuum manifold. On the right is a string defect configuration in real space. Arrows indicate local choices of the phase angle, θ, in the order parameter, Ψ. A complete traversal around the defect region corresponds to a non-trivial winding in S^1. Loci of many such defected regions stacked on top of each other produce an extended string defect (red line).

where m is the mass of He^4.

In the ordered phase, the solution of this equation is

$$\Psi = \sqrt{\frac{\alpha}{\beta}} f(r/\zeta) e^{i\theta}, \tag{9}$$

where $\zeta = \hbar/\sqrt{2m\alpha}$ is the correlation length, and f is a radial function that vanishes at the origin and remain small for radius r smaller than the correlation length.

We see that in the ordered phase the ground state manifold has the topology of a circle, S^1. Regions in the fluid separated by a distance greater that ζ will choose to break the $U(1)$ symmetry of the system arbitrarily, leading to random choices of θ throughout the fluid. It is then possible to find a set of contiguous regions where the phases were chosen in a way that corresponds to a non-trivial winding in the order parameter space. This is illustrated in Fig. 9. The region that is trapped inside remain in the high symmetry $\Psi = 0$ phase, and is known as a vortex, or a string.

The cosmic string defect forms in the same manner. The simplest theory exhibiting a string defect is the Goldstone model [7] in which the Lagrangian is given by

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \Psi^\dagger\right)\left(\partial^\mu \Psi\right) - \frac{\lambda}{4}\left(\Psi^2 - \eta^2\right)^2, \tag{10}$$

where $\mu = 0,1,2,3$ denotes spacetime coordinates, Ψ is some complex scalar field, λ, η are constants, and repeated indices are summed over.

The potential energy term is shaped like a Mexican hat (Fig. 9) and has a $U(1)$ symmetry as raising Ψ to even powers removes the $U(1)$ phase. In the ordered phase the ground state solution is $\Psi = \eta e^{i\theta}$, where θ is the phase angle. One can see that Ψ is not invariant under a $U(1)$ transformation, and has spontaneously broken the full symmetry of the parent system. Between 10^{-43}s and 10^{-35}s after the Big Bang [7], cooling down of the universe purportedly drove the spontaneous symmetry breaking in the Goldstone theory, and causally separated regions in space broke the symmetry (chose θ) in an uncorrelated way. This problem is then completely analogous to the superfluid He^4 case, and following the same arguments as given above one deduces that this theory admits the illusive cosmic string defect.

V. CONCLUSION

In this paper we saw that phase transitions are often associated with a spontaneous symmetry breaking. By defining a suitable order parameter and studying the topology of the order parameter space, topological defects can be predicted and characterized. Because seemingly different systems may have the same underlying symmetrical and topological properties, these defects manifest in all areas of physics - from condensed matter to cosmology.

I leave the reader with a table summarizing the various systems we have studied so far.

System	Topology	Homotopy	Defect
2d Ising	$\mathbb{Z}_2$	$\pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$	Domain wall
2d crystal	T^2	$\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$	Dislocation
$SO(3)$ Higgs	S^2	$\pi_2(S^2) = \mathbb{Z}$	Hedgehog
Superfluid He^4	S^1	$\pi_1(S^1) = \mathbb{Z}$	String
Goldstone	S^1	$\pi_1(S^1) = \mathbb{Z}$	String

TABLE I. A summary of what we have learned. Column 1 names the physical system, column 2 gives the topology of the degenerate ground state manifold in the ordered phase, column 3 gives the lowest degree non-trivial homotopy group, and column 4 names the type of topological defect.

[1] James Sethna, *Statistical mechanics: entropy, order parameters, and complexity*, Vol. 14 (Oxford University Press, 2006).

[2] Ralph Kenna, "Homotopy in statistical physics," arXiv preprint cond-mat/0602459 (2006).

[3] N David Mermin, "The topological theory of defects in ordered media," Reviews of Modern Physics **51**, 591 (1979).

[4] PC Hohenberg and AP Krekhov, "An introduction to the ginzburg–landau theory of phase transitions and nonequilibrium patterns," Physics Reports **572**, 1–42 (2015).

[5] Thomas WB Kibble, "Topology of cosmic domains and strings," Journal of Physics A: Mathematical and General **9**, 1387 (1976).

[6] An introduction to hedgehogs can be found here: https://www.youtube.com/watch?v=-RkaFw0QrRo.

[7] A Vilenkin and EPS Shellard, "Cosmic strings and other topological defects," Cosmic Strings and Other Topological Defects, by A. Vilenkin and EPS Shellard, pp. 537. ISBN 0521391539. Cambridge, UK: Cambridge University Press, January 1995. , 537 (1995).

[8] "Higgs mechanism," http://bolvan.ph.utexas.edu/~vadim/classes/17s/Higgs.pdf.

[9] Alexander M Polyakov, *Particle spectrum in the quantum field theory*, Tech. Rep. (SIS-74-4677, 1974).

[10] Gerard Hooft, "Magnetic monopoles in unified gauge theories," Nuclear Physics: B **79**, 276–284 (1974).

[11] Jeff Paul Greensite and Stefan Olejník, *Confinement, Topology, and Other Non-Perturbative Aspects of QCD*, Vol. 83 (Springer Science & Business Media, 2012).

[12] PC Hendry, N Stuart Lawson, RAM Lee, Peter VE McClintock, and CDH Williams, "Generation of defects in superfluid he-4 as an analogue of the formation of cosmic strings." Nature **368**, 315–317 (1994).

[13] Wojciech H Zurek, "Cosmological experiments in superfluid helium?" Nature **317**, 505–508 (1985).

[14] The Mexican hat potential figure was taken from https://commons.wikimedia.org/wiki/File%3AMexican_hat_potential_polar.svg.