Topological defects in condensed matter and cosmology

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Overview

- Symmetry
- Phase transition, spontaneous symmetry breaking and order
- Topology of order parameter space
- Homotopy
- Topological defects
Learning objectives

• Understand the connection between symmetry and order
• Understand the connection between topology and defects
• Appreciate homotopy theory as a natural language for thinking about topological defects
What is symmetry?

- Pop quiz:
  - Which has more symmetry, ice or water?
What is symmetry?

- Pop quiz:
  - Which has more symmetry, ice or water?
- Answer:
  - *Water!* It has continuous translational and rotational symmetry. Ice breaks both symmetries.
Symmetry breaking phase transition

- During water to ice phase transition the continuous translation and rotation symmetries are broken.
- Water is a high symmetry, disordered phase
- Ice is a low symmetry, ordered phase
- The orientation of the bonds in ice is arbitrarily chosen – the symmetry breaking is spontaneous.
Another example: Ising model

\[ \mathcal{H}_{\text{Ising}} = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z \]

- Ising Hamiltonian is invariant under a global spin-flip
  \[ \sigma_i^z \rightarrow -\sigma_i^z, \forall i \]
- In the disordered phase the average magnetization per site vanishes
  \[ \bar{\sigma} = 0 \]
  \( \bar{\sigma} \) is as symmetric as the Hamiltonian
- In the ferromagnetic phase
  \[ \bar{\sigma} = +1 \text{ or, } -1 \]
  \( \bar{\sigma} \) has less symmetry compared to the Hamiltonian

spontaneous symmetry breaking during phase transition
What is the order parameter?

- Order parameter characterizes the phase
  - There is crystalline order in ice, but none in liquid
  - Magnetization is finite in the ferromagnetic phase, but zero in the paramagnetic phase
- Order breaks the full symmetry of the system
Topology of the order parameter space

• In the ordered phase the degeneracy of the ground states gives rise to a manifold of the order parameter

• Example: Ising Model; order parameter - magnetization
Topology of the order parameter space

- In the ordered phase the degeneracy of the ground states gives rise to a manifold of the order parameter
- Example: 2d crystal; order parameter – atomic displacement

\[ \mathbf{d}(\mathbf{r}) \equiv \mathbf{d}(\mathbf{r}) + ma\hat{x} + na\hat{y} \]
Topology of the order parameter space

- In the ordered phase the degeneracy of the ground states gives rise to a manifold of the order parameter
- **Example:** 3d ferromagnet; order parameter – magnetization vector
Mathematical intermission: what is topology?

- Topology is a classification of space
- Two spaces that can be smoothly deformed into one another are topologically equivalent. Examples:
  - European and American football, $S^2$
  - Donut and coffee cup, $T^2$
  - Circle and square, $S^1$
Mathematical intermission: what is homotopy?

- Homotopy is a classification of the type of closed loops that can be drawn on a topological space.
Mathematical intermission: what is a homotopy group?

- The set of all classes of N-dimensional loops on a topological space forms a group, $\pi_N$
- Example: $S^2$
  - Connected space $\rightarrow \pi_0(S^2) = \mathbb{I}$
  - Any 1d loop can be shrunk to a point $\rightarrow \pi_1(S^2) = \mathbb{I}$
  - 2d “loops” can be used to wrap the surface $\rightarrow \pi_2(S^2) = \mathbb{Z}$
Why do I care about homotopy group?

- Homotopy group predicts and characterizes topological defects
Quick recap

- Phase transition, spontaneous symmetry breaking and order are related
- Order parameter lives on a topological space
- Homotopy classifies the type of N-dimensional loops that can be drawn on a topological space
Topological defect: domain wall

- Ising model in Landau-Ginzburg formalism with order parameter $\phi$
- Ordered ground state is 2-fold degenerate
- Kibble’s argument: causally disconnected regions will choose ground state independently
- Region where two degenerate phases continuously connect is a high energy phase → domain wall
- This defect is connected to the topology of the order parameter space
- In this case the zeroth homotopy group is non-trivial

$$F(\phi, t) = \frac{1}{2} a_1(t) \phi^2 + \frac{1}{4} a_2(t) \phi^4$$
Topological defect: dislocation

- A dislocation defect is a region with extra rows or columns of atoms
- Order parameter space is a 2-torus
- Traversing the defect in real space produces loops around/through the hole of the 2-torus
- First homotopy group is non-trivial

\[ \pi_1(T^2) = \mathbb{Z} \times \mathbb{Z} \]
Let’s pause a little...

- There is always a non-trivial homotopy group for every topological defect
- We can first analyze the topology, predict the defect, and then look for it
Topological defect for 2-sphere?

- Pop quiz:
  - Is there a topological defect for an order parameter space with the topology of a 2-sphere?
Topological defect for 2-sphere?

- **Pop quiz:**
  - Is there a topological defect for an order parameter space with the topology of a 2-sphere?

- **Answer:**
  - *Yes!* Because the second homotopy group is non-trivial.

\[ \pi_2(S^2) = \mathbb{Z} \]
Topological defect: hedgehog

\[ \mathcal{H} = \int d^3 x \left\{ \frac{1}{2} \pi_a^2 + \frac{1}{2} (\nabla \phi_a)^2 - \frac{\mu^2}{2} \phi_a^2 + \frac{\lambda}{4} \phi_a^4 \right\} \]

- Higgs real triplet \( \phi_a \)
  - Hamiltonian has SO(3) rotation symmetry
- In the ordered phase "z"-component of the Higgs triplet is fixed, the other two are zero
  - \( \phi_a \) has SO(2) symmetry
  - \( \phi_a \) space is a 2-sphere
- Solution: \( \phi_a \) is a radial vector in real space
  - spiky, like a hedgehog
  - Covering the hedgehog means non-trivially wrapping the 2-sphere

\[ \pi_2(S^2) = \mathbb{Z} \]
Topological defect for a circle?

- Pop quiz:
  - Is there a topological defect for an order parameter space with the topology of a circle?
Topological defect for a circle?

• Pop quiz:
  – Is there a topological defect for an order parameter space with the topology of a circle?

• Answer:
  – Yes! Because the first homotopy group is non-trivial.

\[ \pi_1(S^1) = \mathbb{Z} \]
**Topological defect: string**

\[ \mathcal{L} = \frac{1}{2} (\partial_{\mu} \Psi^\dagger) (\partial^\mu \Psi) - \frac{\lambda}{4} (\Psi^2 - \eta^2)^2 \]

- Goldstone model
  - \( \Psi \) is complex
  - Hamiltonian has U(1) symmetry
- In the ordered phase
  - phase angle is chosen arbitrarily
  - topology of phase angle space is a circle
  - contiguous regions exist such that a full twist of phase angle is made
  - at the center of this region resides a high symmetry phase

\[ \pi_1(S^1) = \mathbb{Z} \]
## Conclusion

- Saw the connection between symmetry and order
- Saw the connection between topology and defects
- Used homotopy theory to characterize and predict topological defects
- Saw that topological defects are generic creatures and can pop up in condensed matter and cosmology alike

<table>
<thead>
<tr>
<th>System</th>
<th>Topology</th>
<th>Homotopy</th>
<th>Defect</th>
</tr>
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<tr>
<td>2d Ising</td>
<td>$\mathbb{Z}_2$</td>
<td>$\pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$</td>
<td>Domain wall</td>
</tr>
<tr>
<td>2d crystal</td>
<td>$T^2$</td>
<td>$\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$</td>
<td>Dislocation</td>
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<td>$SO(3)$ Higgs</td>
<td>$S^2$</td>
<td>$\pi_2(S^2) = \mathbb{Z}$</td>
<td>Hedgehog</td>
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<tr>
<td>Goldstone</td>
<td>$S^1$</td>
<td>$\pi_1(S^1) = \mathbb{Z}$</td>
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