Topological defects in condensed matter and cosmology

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Overview

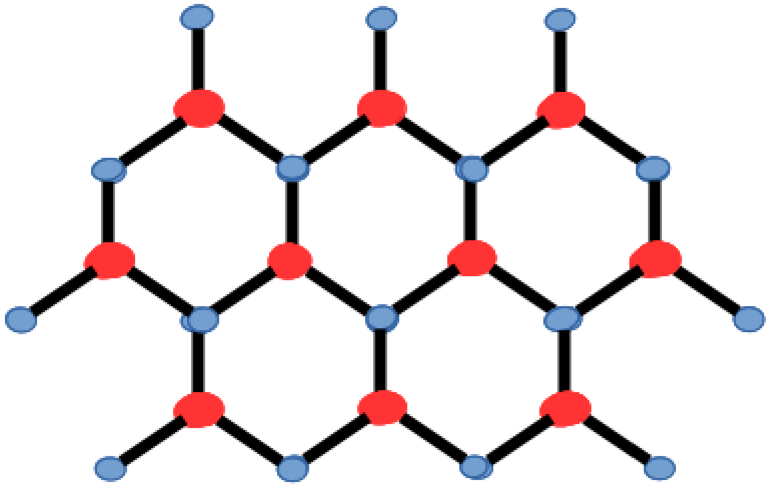
- Symmetry
- Phase transition, spontaneous symmetry breaking and order
- Topology of order parameter space
- Homotopy
- Topological defects

Learning objectives

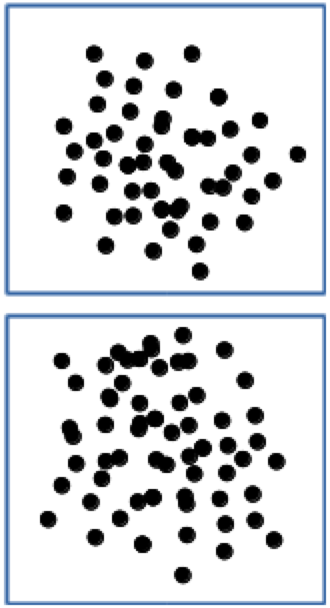
- Understand the connection between symmetry and order
- Understand the connection between topology and defects
- Appreciate homotopy theory as a natural language for thinking about topological defects

What is symmetry?

- Pop quiz:
  - Which has more symmetry, ice or water?



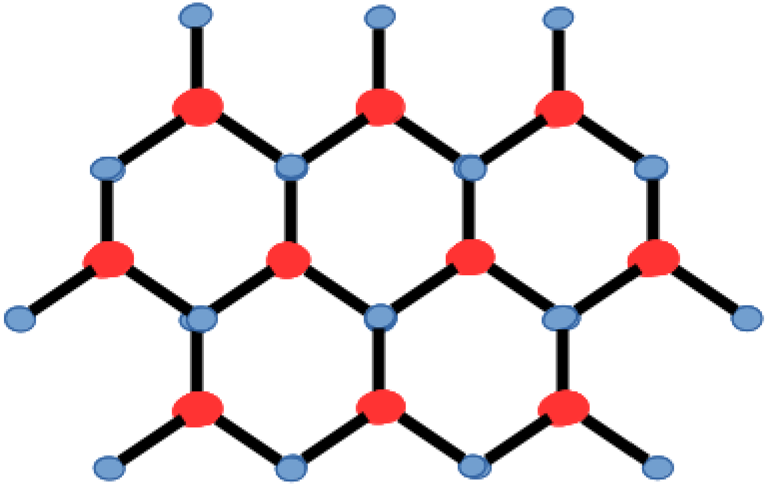
Ice



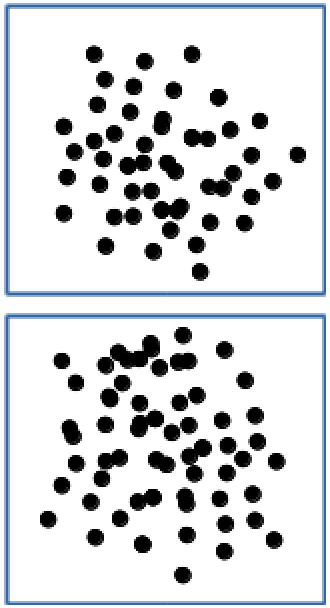
Water

What is symmetry?

- Pop quiz:
  - Which has more symmetry, ice or water?
- Answer:
  - *Water!* It has continuous translational and rotational symmetry. Ice breaks both symmetries.



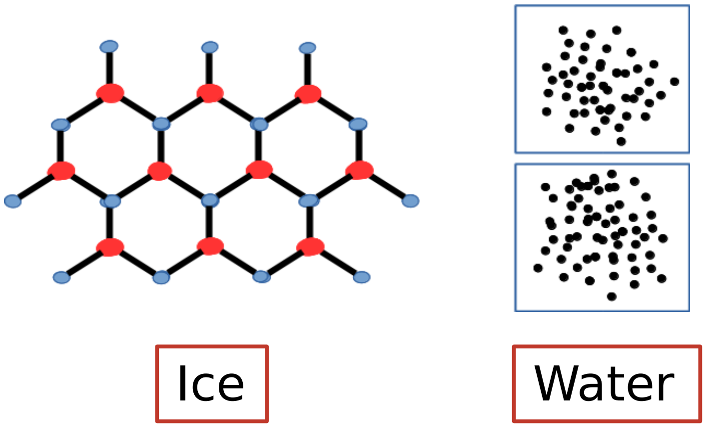
Ice



Water

Symmetry breaking phase transition

- During water to ice phase transition the continuous translation and rotation symmetries are broken.
- Water is a high symmetry, disordered phase
- Ice is a low symmetry, ordered phase
- The orientation of the bonds in ice is arbitrarily chosen – the symmetry breaking is spontaneous.



Another example: Ising model

$$\mathcal{H}_{\text{Ising}} = -J \sum_{<i,j>} \sigma_i^z \sigma_j^z$$

- Ising Hamiltonian is invariant under a global spin-flip

$$\sigma_i^z \to -\sigma_i^z, \forall i$$

- In the disordered phase the average magnetization per site vanishes

$$\bar{\sigma} = 0$$

$\bar{\sigma}$ is as symmetric as the Hamiltonian

- In the ferromagnetic phase

$$\bar{\sigma} = +1 \text{ or, } -1$$

$\bar{\sigma}$ has less symmetry compared to the Hamiltonian

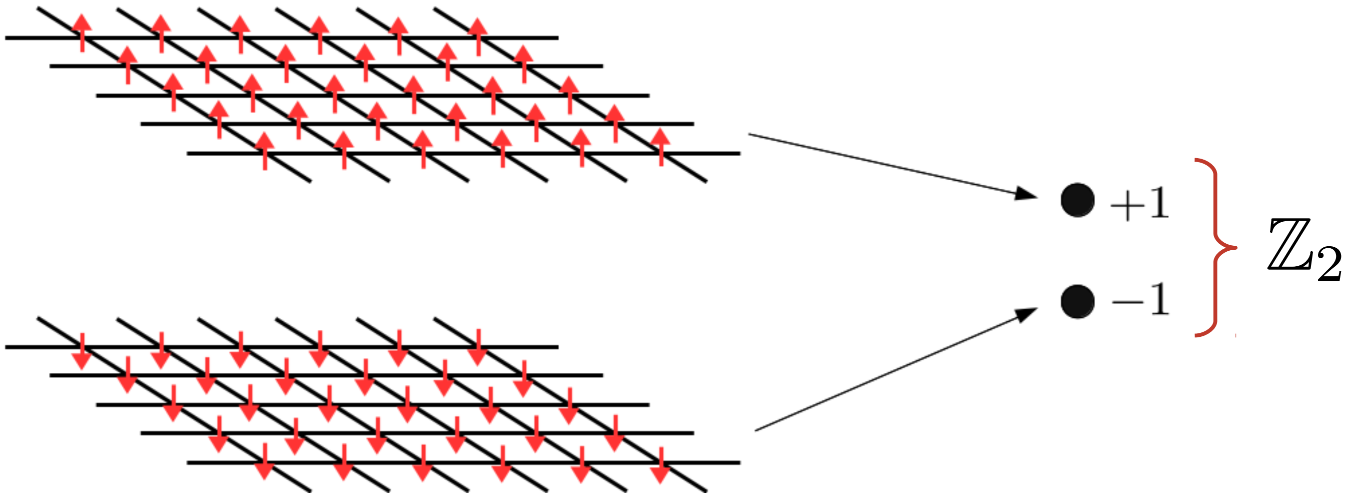
spontaneous symmetry breaking during phase transition

What is the order parameter?

- Order parameter characterizes the phase
  - There is crystalline order in ice, but none in liquid
  - Magnetization is finite in the ferromagnetic phase, but zero in the paramagnetic phase
- Order breaks the full symmetry of the system

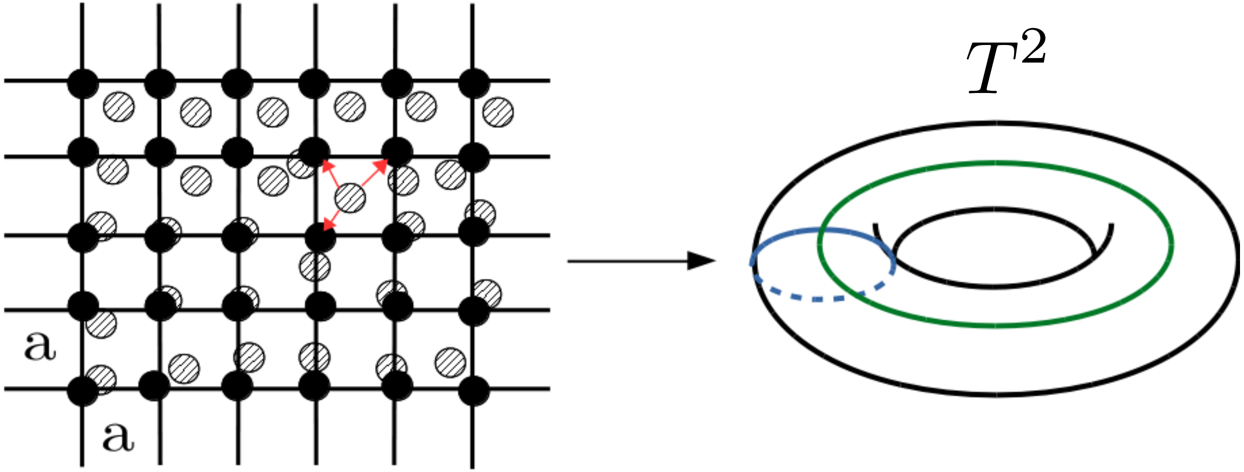
Topology of the order parameter space

- In the ordered phase the degeneracy of the ground states gives rise to a manifold of the order parameter
- Example: Ising Model; order parameter - magnetization



Topology of the order parameter space

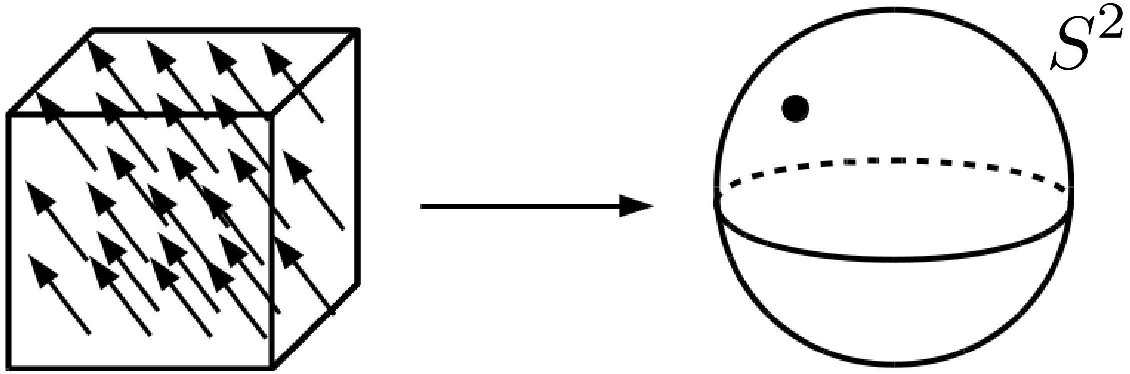
- In the ordered phase the degeneracy of the ground states gives rise to a manifold of the order parameter
- Example: 2d crystal; order parameter – atomic displacement



$$\mathbf{d}(\mathbf{r}) \equiv \mathbf{d}(\mathbf{r}) + ma\hat{x} + na\hat{y}$$

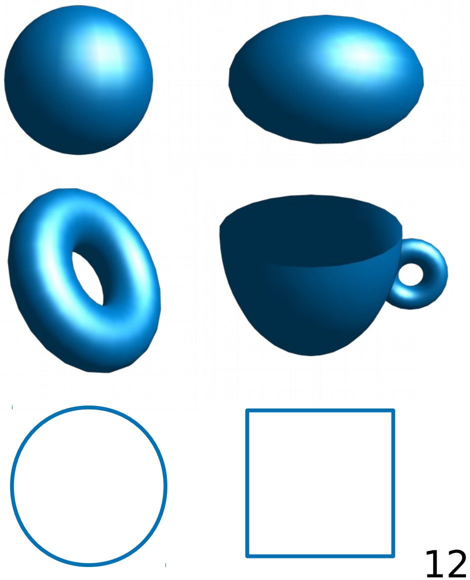
Topology of the order parameter space

- In the ordered phase the degeneracy of the ground states gives rise to a manifold of the order parameter
- Example: 3d ferromagnet; order parameter – magnetization vector



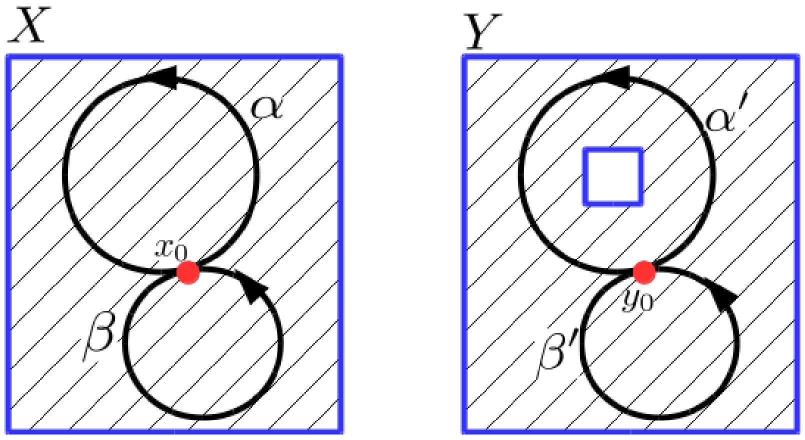
Mathematical intermission: what is topology?

- Topology is a classification of space
- Two spaces that can be smoothly deformed into one another are topologically equivalent. Examples:
  - European and American football, S^2
  - Donut and coffee cup, T^2
  - Circle and square, S^1



Mathematical intermission: what is homotopy?

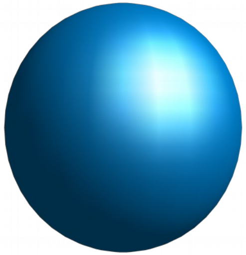
- Homotopy is a classification of the type of closed loops that can be drawn on a topological space



$\alpha \sim \beta$ $\qquad$ $\alpha' \nsim \beta'$

Mathematical intermission: what is a homotopy group?

- The set of all classes of N-dimensional loops on a topological space forms a group, π_N
- Example: S^2
  - Connected space → $\pi_0(S^2) = \mathbb{I}$
  - Any 1d loop can be shrunk to a point → $\pi_1(S^2) = \mathbb{I}$
  - 2d “loops” can be used to wrap the surface → $\pi_2(S^2) = \mathbb{Z}$



Why do I care about homotopy group?

- Homotopy group predicts and characterizes topological defects

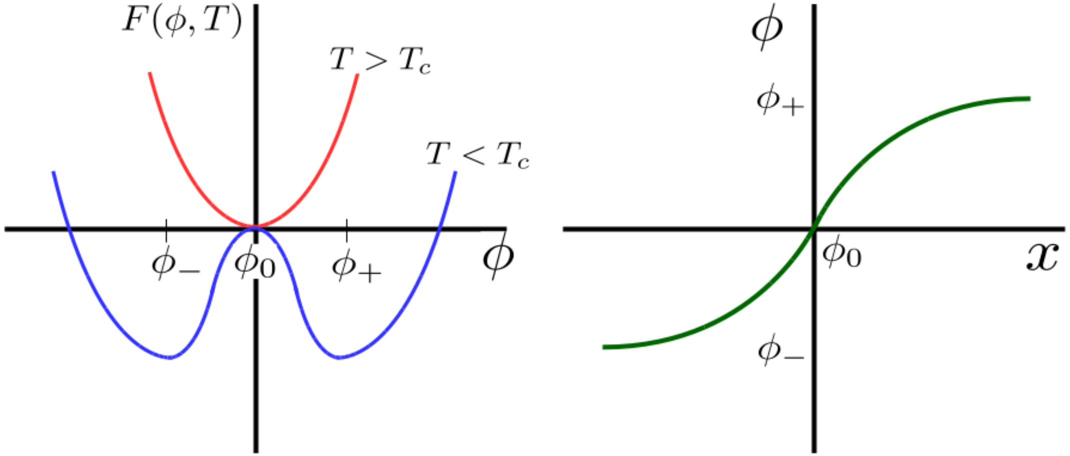
Quick recap

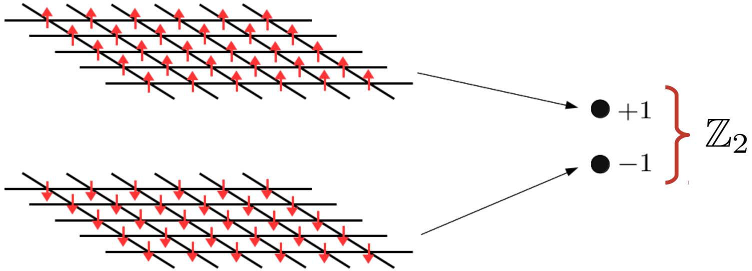
- Phase transition, spontaneous symmetry breaking and order are related
- Order parameter lives on a topological space
- Homotopy classifies the type of N-dimensional loops that can be drawn on a topological space

Topological defect: domain wall

$$F(\phi, t) = \frac{1}{2}a_1(t)\phi^2 + \frac{1}{4}a_2(t)\phi^4$$

- Ising model in Landau-Ginzburg formalism with order parameter ϕ
- Ordered ground state is 2-fold degenerate
- Kibble's argument: causally disconnected regions will choose ground state independently
- Region where two degenerate phases continuously connect is a high energy phase → domain wall
- This defect is connected to the topology of the order parameter space
- In this case the zeroth homotopy group is non-trivial

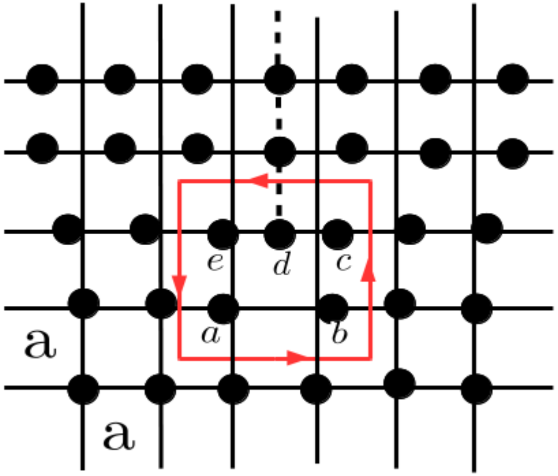


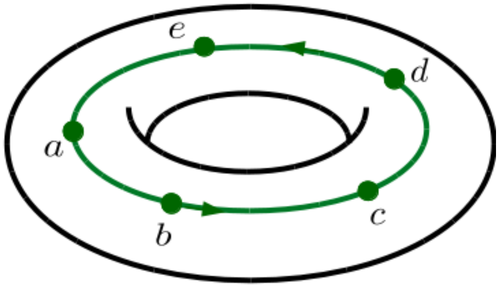


$$\pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$$

Topological defect: dislocation

- A dislocation defect is a region with extra rows or columns of atoms
- Order parameter space is a 2-torus
- Traversing the defect in real space produces loops around/through the hole of the 2-torus
- First homotopy group is non-trivial





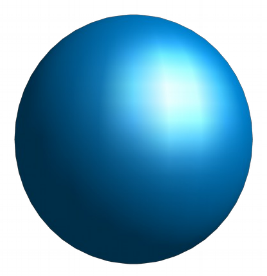
$$\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$$

Let’s pause a little...

- There is always a non-trivial homotopy group for every topological defect
- We can first analyze the topology, predict the defect, and then look for it

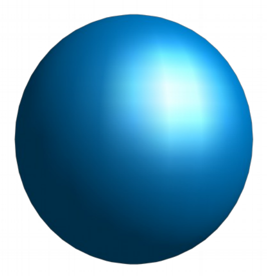
Topological defect for 2-sphere?

- Pop quiz:
  - Is there a topological defect for an order parameter space with the topology of a 2-sphere?



Topological defect for 2-sphere?

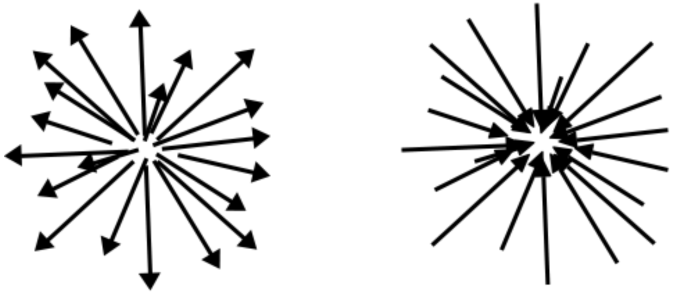
- Pop quiz:
  - Is there a topological defect for an order parameter space with the topology of a 2-sphere?
- Answer:
  - *Yes!* Because the second homotopy group is non-trivial.



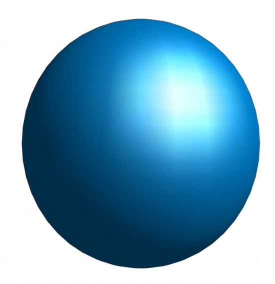
$$\pi_2(S^2) = \mathbb{Z}$$

Topological defect: hedgehog

$$\mathcal{H} = \int \mathrm{d}^3x \left\{ \frac{1}{2}\pi_a^2 + \frac{1}{2}\left(\nabla\phi_a\right)^2 - \frac{\mu^2}{2}\phi_a^2 + \frac{\lambda}{4}\phi_a^4 \right\}$$



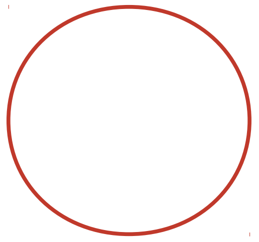
- Higgs real triplet ϕ_a
  - Hamiltonian has SO(3) rotation symmetry
- In the ordered phase “z”-component of the Higgs triplet is fixed, the other two are zero
  - ϕ_a has SO(2) symmetry
  - ϕ_a space is a 2-sphere
- Solution: ϕ_a is a radial vector in real space
  - spiky, like a hedgehog
  - Covering the hedgehog means non-trivially wrapping the 2-sphere



$$\pi_2(S^2) = \mathbb{Z}$$

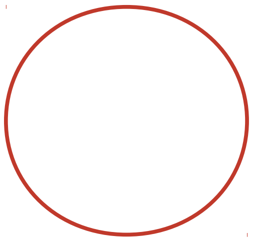
Topological defect for a circle?

- Pop quiz:
  - Is there a topological defect for an order parameter space with the topology of a circle?



Topological defect for a circle?

- Pop quiz:
  - Is there a topological defect for an order parameter space with the topology of a circle?
- Answer:
  - *Yes!* Because the first homotopy group is non-trivial.

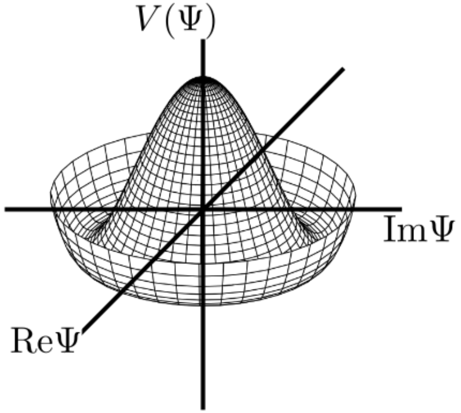


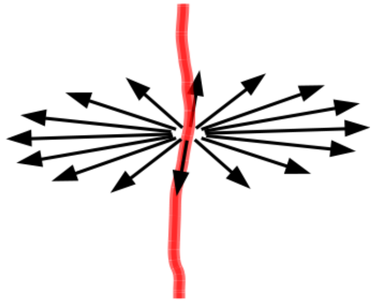
$$\pi_1(S^1) = \mathbb{Z}$$

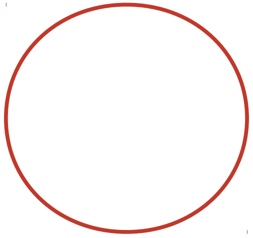
Topological defect: string

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \Psi^\dagger\right)\left(\partial^\mu \Psi\right) - \frac{\lambda}{4}\left(\Psi^2 - \eta^2\right)^2$$

- Goldstone model
  - Ψ is complex
  - Hamiltonian has U(1) symmetry
- In the ordered phase
  - phase angle is chosen arbitrarily
  - topology of phase angle space is a circle
  - contiguous regions exist such that a full twist of phase angle is made
  - at the center of this region resides a high symmetry phase







$$\pi_1(S^1) = \mathbb{Z}$$

Conclusion

- Saw the connection between symmetry and order
- Saw the connection between topology and defects
- Used homotopy theory to characterize and predict topological defects
- Saw that topological defects are generic creatures and can pop up in condensed matter and cosmology alike

System	Topology	Homotopy	Defect
2d Ising	$\mathbb{Z}_2$	$\pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$	Domain wall
2d crystal	T^2	$\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$	Dislocation
$SO(3)$ Higgs	S^2	$\pi_2(S^2) = \mathbb{Z}$	Hedgehog
Goldstone	S^1	$\pi_1(S^1) = \mathbb{Z}$	String