Memory-less Processes for Discrete Events, and Their Distributions

	Number of events or successes in a period.	Waiting Time for the next event or The time between consecutive events.	Waiting time for the r-th event
Discrete Time	**Binomial Distribution** $X \sim Bin(n, p)$ $P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$, $k = 0,1,..$ n = (# of trials), period = (n trials) p = (success probability on one trial) $E(X) = np$; Var $(X) = n\,p\,(1-p)$ $M_X(t)=(1-p+pe^t)^n$ $\sum_{\text{Ind}} \text{Bin}(n_i,p) \sim \text{Bin}(\sum n_i, p)$	**Geometric Distribution** $X \sim Geom(p)$ $P(X=k)=(1-p)^{k-1}p$, $k = 1, 2, ..$ k = (# trials until the next success) $E(X)=\frac{1}{p}$; $Var(X)=\frac{1-p}{p^2}$ $M_X(t)=\frac{pe^t}{1-(1-p)e^t}$ Independent sums are not geometric. Ind. sums w/ same p are Neg. Binomial.	**Negative Binomial Distribution** $X \sim NgB(p, r)$ $P(X=k)=\binom{k-1}{r-1}p^r(1-p)^{k-r}$, $k = r, r+1, ..$ $E(X)=\frac{r}{p}$; $Var(X)=\frac{r(1-p)}{p^2}$ $M_X(t)=(\frac{pe^t}{1-(1-p)e^t})^r$ $\sum_{\text{Ind}} NgB(p,r_i) \sim NgB(p,\sum r_i)$
Continuous Time	**Poisson Distribution** $X \sim Pois(\lambda)$ $P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$, $k=0,1,...$ λ = (average event rate per unit time) period = (unit time $\Delta T = 1$) $E(X) = \lambda$; Var $(X) = \lambda$ $M_X(t)=e^{\lambda e^t-\lambda}$ $\sum_{\text{Ind}} \text{Pois}(\lambda_i) \sim \text{Pois}(\sum \lambda_i)$	**Exponential Distribution** $X \sim Exp(\lambda)$] $f_X(x)=\lambda e^{-\lambda x}$, $x>0$ $E(X)=\frac{1}{\lambda}$; $Var(X)=\frac{1}{\lambda^2}$ $M_X(t)=\frac{\lambda}{\lambda-t}$ Independent sums are not exponential. Ind. sums with same λ are Gamma. $X \sim Exp(\lambda) \Rightarrow aX \sim Exp(\frac{\lambda}{a})$	**Gamma Distribution** $X \sim Gam(\lambda, r)$ $f_X(x)=\frac{\lambda^r}{(r-1)!}x^{r-1}e^{-\lambda x}$, $x>0$ $E(X)=\frac{r}{\lambda}$; $Var(X)=\frac{r}{\lambda^2}$ $M_X(t)=(\frac{\lambda}{\lambda-t})^r$ $\sum_{\text{Ind}} Gam(\lambda,r_i) \sim Gam(\lambda,\sum r_i)$ (can be generalized to real $r > 0$ using the gamma function for $(r-1)!$)
Connections Between Discrete and Continuous Time	$0 \le p \le 1$, $n > 0$, $\lambda > 0$ $np = \lambda$ = (average # events per period) $\lim_{n\to\infty} P(Bin(n,\frac{\lambda}{n})=k)=P(Pois(\lambda)=k)$	Memory-less-ness, for these two, says: $P(X \le a+b \mid X>a)=P(X\le b)$ $\frac{k}{n}=\frac{kp}{\lambda}=x=$ (# periods until next event) $\lim_{p\to 0} F_{Geom(p)}(\frac{x\lambda}{p})=F_{Exp(\lambda)}(x)$	Each of these is an independent sum of the RV's to its left in the chart. $\lim_{p\to 0} F_{NgB(p,r)}(\frac{x\lambda}{p})=F_{Gam(\lambda,r)}(x)$