A Proof of the Central Limit Theorem.

Notation. Let $X_1,\ldots,X_n$ be RV’s.

Let $\sum_n X$ denote the sum $X_1 + \ldots + X_n$, and let $\overline{X}_n$ denote the average $\left(\sum_n X\right)/n$.

When there is no ambiguity, we drop the subscript and write $\sum X$ or $\overline{X}$.

Lemma 1 (results we have shown)

Let $X_1,\ldots,X_n$ be IID with common mean $\mu$ and variance $\sigma^2$ ($\sigma > 0$), and let $Y_i = \frac{X_i - \mu}{\sigma}$. Then

a) $Y_1,\ldots,Y_n$ are IID with common mean 0 and variance 1 (and hence standard deviation 1)

b) $M_{Y_i}^\prime(0) = 0$ (since $M_{Y_i}^\prime(0) = E(Y_i)$)

c) $W_n = \frac{1}{\sqrt{n}} \sum_n Y$ has mean $\frac{1}{\sqrt{n}} \sum E(Y_i) = 0$ and variance $(\frac{1}{\sqrt{n}})^2 \sum \text{Var}(Y_i) = \frac{1}{n}(n) = 1$.

Lemma 2. (results from calculus)

a) Let $f^\prime(t)$ be continuous for $-c < t < c$. Then, for all such $t$ between $-c$ and $c$,

$$f(t) = f(0) + f'(0)t + \frac{t^2}{2!} f''(r) \text{ for some } r \text{ with } |r| < |t|.$$  

b) Let $g$ be a function such that $\lim_{n \to \infty} g(n) = B$. Then $\lim_{n \to \infty} \left(1 + \frac{g(n)}{n}\right)^n = e^B$.

proof. : (a) is just Taylor’s Theorem for $a = 0$ [aka McLaurin series] and $n = 1$.

b) Let $L = \lim_{n \to \infty} \left(1 + \frac{g(n)}{n}\right)^n$. Let $h$ denote $\frac{g(n)}{n}$, i.e. $h = \frac{g(n)}{n}$.

Then $\ln(L) = \lim_{n \to \infty} \ln(1 + \frac{g(n)}{n}) = \lim_{n \to \infty} g(n) \frac{\ln(1 + \frac{g(n)}{n})}{g(n)} = \lim_{n \to \infty} \ln(1 + \frac{g(n)}{n}) = \lim_{n \to \infty} \ln(1 + \frac{g(n)}{n}) = \lim_{n \to \infty} \ln(1 + h) / h = B\lim_{h \to 0} \ln(1 + h) / h = B\ln'(1) = B(1) = B$. Thus $L = e^B$.

Lemma 3. If $W_1, W_2, \ldots$ are RV’s such that for all $t$ in some open interval around 0,

$$\lim_{n \to \infty} M_{W_n}(t) = M_W(t), \text{ then for all real numbers } b, \lim_{n \to \infty} F_{W_n}(b) = F_W(b).$$

proof. This result requires more real analysis than we have available, so we’ll have to accept this result.
The Central Limit Theorem.

Let \( X_1, X_2, \ldots \) be any infinite sequence of IID RV’s with (common) mean \( \mu \) and variance \( \sigma^2 \). Let \( Z \) be a standard, normal RV, i.e. \( Z \sim \text{Norm}(0,1) \). Then, for all real numbers \( b \),

\[
\begin{align*}
\text{[Sum version]} \quad & \lim_{n \to \infty} P\left( \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} < b \right) = P(Z < b), \text{ i.e. } \lim_{n \to \infty} F_{\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma}}(b) = F_Z(b), \text{ or equivalently,} \\
\text{[Mean version]} \quad & \lim_{n \to \infty} P\left( \frac{\bar{X}_n - \mu}{(\sigma / \sqrt{n})} < b \right) = P(Z < b), \text{ i.e. } \lim_{n \to \infty} F_{\frac{\bar{X}_n - \mu}{(\sigma / \sqrt{n})}}(b) = F_Z(b). \text{ in other words,}
\end{align*}
\]

The random variable

\[
\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} = \frac{\bar{X}_n - \mu}{(\sigma / \sqrt{n})}
\]

approaches a standard normal distribution as \( n \to \infty \).

proof. We give the proof only for \( X \)'s for which the mgf converges in some neighborhood of 0, in particular, all moments of \( X \) are finite. The theorem holds as stated, but the more general proof involves complex analysis.

First define \( Y_i = \frac{X_i - \mu}{\sigma} \), then \( Y_1, Y_2, \ldots \) are IID with mean 0 and variance 1.

By Lemma 1b and 2a, \( M_Y(t) = 1 + 0t + \frac{t^2}{2!} M_Y''(r) = 1 + \frac{t^2}{2!} M_Y''(r) \) with \( r = r(t) \) satisfying \( |r| < |t| \).

Let \( W_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i \); then \( M_{W_n}(t) = M_Y\left(\frac{t}{\sqrt{n}}\right)^n = (1 + \frac{t^2}{2n} M_Y''(s))^n \) with \( s = r\left(\frac{t}{\sqrt{n}}\right) \Rightarrow s| < \frac{|t|}{\sqrt{n}} \).

As \( n \to \infty \), then \( s \to 0 \) and so by continuity,

\[
\lim_{n \to \infty} M_Y''(s) = M_Y''(0) = E(Y^2) = \text{Var}(Y) + E(Y)^2 = 1 + 0^2 = 1
\]

Let \( g(n) = \frac{t^2}{2} M_Y''\left(\frac{t}{\sqrt{n}}\right) \), then \( \lim_{n \to \infty} g(n) = \frac{t^2}{2} (1) = \frac{t^2}{2} \).

By Lemma 2b, \( \lim_{n \to \infty} M_{W_n}(t) = \lim_{n \to \infty} \left(1 + \frac{g(n)}{n}\right)^n = e^{\frac{t^2}{2}} = M_Z(t) \) where \( Z \sim \text{Norm}(0,1) \).

By Lemma 3, \( \lim_{n \to \infty} F_{W_n}(b) = F_Z(b) \), i.e.

\[
P(Z < b) = \lim_{n \to \infty} P(W_n < b) = \lim_{n \to \infty} P\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i < b\right) = \lim_{n \to \infty} P\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right) < b\right) = \lim_{n \to \infty} P\left(\frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma \sqrt{n}} < b\right)
\]

QED.