AIP | Chaos
An Interdisciplinary Journal of Nonlinear Science

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Citation: Chaos: An Interdisciplinary Journal of Nonlinear Science **22**, 043118 (2012); doi: 10.1063/1.4766596
View online: http://dx.doi.org/10.1063/1.4766596
View Table of Contents: http://scitation.aip.org/content/aip/journal/chaos/22/4?ver=pdfcov
Published by the AIP Publishing

# The asymptotic behavior of the order parameter for the infinite-$N$ Kuramoto model

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(Received 28 February 2012; accepted 22 October 2012; published online 26 November 2012)

The Kuramoto model, first proposed in 1975, consists of a population of sinusoidally coupled oscillators with random natural frequencies. It has served as an idealized model for coupled oscillator systems in physics, chemistry, and biology. This paper addresses a long-standing problem about the infinite-$N$ Kuramoto model, which is to describe the asymptotic behavior of the order parameter for this system. For coupling below a critical level, Kuramoto predicted that the order parameter would decay to 0. We use Fourier transform methods to prove that for general initial conditions, this decay is not exponential; in fact, exponential decay to 0 can only occur if the initial condition satisfies a fairly strong regularity condition that we describe. Our theorem is a partial converse to the recent results of Ott and Antonsen, who proved that for a special class of initial conditions, the order parameter does converge exponentially to its limiting value. Consequently, our result shows that the Ott–Antonsen ansatz does not completely capture all the possible asymptotic behavior in the full Kuramoto system. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4766596]

**The Kuramoto coupled oscillator model, first introduced by Kuramoto almost forty years ago, has served as a paradigm for coupled oscillator systems in biology and physics. This paper addresses a long-standing problem in the study of the Kuramoto model, which is to characterize the rate at which of the system's order parameter approaches its limiting value. Recently, Ott and Antonsen proved some important results in this direction, establishing exponential convergence of the order parameter for a specialized set of initial conditions. The main result in this work is in some ways a converse to theirs; I prove that for general initial conditions, exponential convergence does not occur. The proof uses Fourier-theoretic methods, Möbius maps, and the theory of Riccati equations, and is the first time the combination of these techniques has been used to analyze the Kuramoto model.**

## I. INTRODUCTION

Kuramoto's famous coupled oscillator model is the system

$$\dot{\theta}_j = \omega_j + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j), \tag{1}$$

where $K \geq 0$ is a constant and the natural frequencies $\omega_j$ are chosen according to a probability density function $g$, which we assume is symmetric about its mean $\Omega$, unimodal and has infinite support. Using a change of coordinates to a rotating frame with angular velocity $\Omega$, we can reduce to the case where the mean $\Omega = 0$; henceforth, we will make this assumption. It is natural to measure the degree to which this system is synchronized at any given moment $t$ by the size of the complex order parameter

$$Z(t) = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}.$$

We have $|Z(t)| = 1$ if (and only if) the system is completely synchronized, meaning that all the phases $\theta_j(t)$ are identical at time $t$. On the other hand, $|Z(t)| \approx 0$ means that the system is at least somewhat unsynchronized; the phases $\theta_j(t)$ cannot all be close to each other. Kuramoto gave a beautiful heuristic calculation[1] which for large $N$ predicts the long-term behavior of $Z(t)$: for $K$ below a critical coupling $K_c$, $Z(t)$ converges to 0; whereas for $K > K_c$, $Z(t)$ converges to some non-zero point inside the unit circle. (The dependence of $r = \lim_{t\to\infty}|Z(t)|$ on $K$ is shown nicely in Fig. 3 in Ref. 2.) Kuramoto even found $K_c$ exactly: $K_c = 2/\pi g(0)$. Numerical simulations of Kuramoto's system support the accuracy of his prediction.

Over the years, much work has been done to try to make some rigorous sense out of Kuramoto's conjecture (see Ref. 2 for a survey of some of this work through 2000). The reason why this problem is so difficult is that the system (1) does not have any fixed points if $|\omega_j| > K$ for some $j$, and this will surely be the case when $N$ is large since the density $g$ has infinite support. Thus, one cannot attack Kuramoto's conjecture by the usual method of linearizing at a fixed point and proving stability from the linearization. One way to get around this difficulty is to study an "infinite-$N$" analog of Eq. (1), which was first proposed by the author and Strogatz in Ref. 3; instead of finitely many points on the unit circle, we defined a state of the infinite-$N$ model to be a family $\rho_\omega$ of densities on the circle, one for each frequency $\omega \in \mathbb{R}$, with a mild regularity condition on the map $\omega \mapsto \rho_\omega$ (we will discuss the topology on this state space below; a very

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detailed description is given in Ref. 4). The order parameter $Z$ for the state $\rho_\omega$ is defined as the average center of mass of $\rho_\omega$ with respect to the density $g(\omega)$

$$Z = Re^{i\psi} = \int_{-\infty}^{\infty}\left(\int_0^{2\pi} e^{i\theta}\rho_\omega(\theta)\,d\theta\right)g(\omega)\,d\omega.$$

Let $v_\omega$ be the vector field on the circle defined by

$$v_\omega(\theta) = \omega + K\,\mathrm{Im}(Ze^{-i\theta}) = \omega + KR\sin(\psi-\theta).$$

Then the evolution of the densities in time is naturally given by the continuity equation

$$\frac{d\rho_\omega}{dt} + \frac{\partial}{\partial\theta}(v_\omega\rho_\omega) = 0. \qquad (2)$$

The advantage of this infinite-$N$ model is that it has genuine fixed states which correspond to the large-$N$ behavior conjectured by Kuramoto. So, one can attack Kuramoto's conjecture by linearizing the system at these states and studying their stability, as was done by the author and Strogatz in Ref. 4. Unfortunately, the conclusion of this analysis is that the fixed states of the infinite-$N$ Kuramoto model are *never* linearly asymptotically stable; they are at best neutrally stable, with continuous spectrum the entire imaginary axis!

It turns out that there is a simple group-theoretic explanation for this neutral stability, which applies to the original system, not just its linearizations. To describe this, we need to explain a bit about the topology of the state space for the infinite-$N$ Kuramoto model. The space $Pr(S^1)$ of probability measures on the circle has a natural topology as a subspace of the unit ball in the dual space $C^1(S^1)^*$, where $C^1(S^1)$ is the Banach space of continuously differentiable functions on $S^1$. This topology on $Pr(S^1)$ is metrizable compact Hausdorff, and therefore cannot be weakened or strengthened without losing either compactness or the Hausdorff property. It has the additional natural feature that if one considers the subspace of $Pr(S^1)$ consisting of unit point mass measures (i.e., delta functions) on the circle, then this subspace has the same topology as the circle $S^1$. The state space $\mathbf{S}$ for the infinite-$N$ Kuramoto model consists of measurable maps $\omega \mapsto \rho_\omega$ from $\mathbb{R}$ to $Pr(S^1)$, with topology inherited from the Banach space $L^\infty(\mathbb{R}, C^1(S^1)^*)$. This is a fairly strong topology on the state space $\mathbf{S}$; in this topology, a sequence of states $\rho^n_\omega$ converges to $\rho_\omega$ if and only if $\rho^n_\omega \to \rho_\omega$ in $Pr(S^1)$ uniformly a.e. in $\omega$.

There is a natural group action on $Pr(S^1)$ of the group $G$ of Möbius transformations which preserve the unit disc. (The important role of this group in the study of the Kuramoto and other related oscillator models was first pointed out by the author and his collaborators in Ref. 5.) Möbius maps are fractional linear transformations on the extended complex plane $\hat{\mathbb{C}}$ given by

$$z \mapsto \frac{az+b}{cz+d}, \quad ad-bc \neq 0,$$

where $a, b, c, d \in \mathbb{C}$ are constants. The Möbius maps which preserve the unit disc have the special form

$$z \mapsto \alpha\frac{z-\beta}{1-\bar{\beta}z},$$

where $|\alpha| = 1$ and $|\beta| < 1$; these Möbius maps form a (real) three-dimensional group $G$. Since $G$ preserves the unit circle $S^1$, the group $G$ acts in a natural way on the space $Pr(S^1)$ of probability measures on $S^1$ via the adjoint formula

$$\int_0^{2\pi} f(\theta)\,d(\phi_*\rho)(\theta) = \int_0^{2\pi}(f\circ\phi)(\theta)\,d\rho(\theta)$$

for $\phi \in G$, $\rho \in Pr(S^1)$ and $f$ any continuous function on $S^1$. It turns out that this action is intimately related to the Kuramoto model: If we denote the time-$t$ flow of a state $\rho_{\omega,0} \in \mathbf{S}$ by $\rho_{\omega,t}$, then for each $\omega$, $\rho_{\omega,t}$ lies in the orbit of $\rho_{\omega,0}$ under the group $G$.[5] So, the dynamics for any given initial condition are constrained to lie on a (relatively) very small submanifold of the state space $\mathbf{S}$. This is an extremely special property of the Kuramoto model, which definitely does not hold for similar models with more general coupling functions. And, as we shall see, it is the basis for the neutral stability in the infinite-$N$ Kuramoto model.

The fixed states $\rho_\omega$ of the infinite-$N$ model are families of a special form of density on the circle, which we call Poisson densities because of the connection to the Poisson integral for harmonic functions. Smooth Poisson densities have the form

$$\rho(\theta) = \frac{a}{1+b\cos\theta+c\sin\theta}, \quad a>0, \quad b^2+c^2<1,$$

where the constants are chosen so that $\int_0^{2\pi}\rho(\theta)\,d\theta = 1$; we also include the limiting case of unit point mass measures on the circle among the Poisson densities. A Poisson density is uniquely determined by its center of mass,[5] so the space of Poisson densities is homeomorphic to the closed unit disc, and hence is compact. Now suppose an initial state $\rho_\omega \in \mathbf{S}$ has non-Poisson densities for some values of $\omega$: then this remains true for all finite time $t$ as the state evolves in the Kuramoto model. What can happen as $t \to \infty$? The orbit of a density $\rho$ under $G$ is not in general compact, but the only points in the closure of the orbit $G\rho$ and not in $G\rho$ are probability measures supported on one or possibly two points. (Roughly speaking, a sequence of Möbius maps in $G$ diverging to $\infty$ pushes more and more of the circle towards one point.) In particular, if $\rho$ is not a Poisson density, then the only Poisson densities in the closure of its orbit under $G$ are unit point mass measures. Now, the fixed states for the infinite-$N$ Kuramoto model consist of smooth Poisson densities for all sufficiently large $\omega$, which therefore can never be reached asymptotically unless the initial conditions for those $\omega$ were Poisson densities to begin with. Therefore, the fixed states for the infinite-$N$ Kuramoto model are not asymptotically stable, so the neutral stability found in Ref. 4 is in retrospect not surprising. In fact, with a bit more fleshing out, this basically gives a much simpler and more conceptual proof of the neutral linear stability established in Ref. 4.

But, the neutral stability of the fixed states does not finish the story. After all, the main object of study in the

Kuramoto model is the order parameter $Z(t)$. How does $Z(t)$ behave as $t \to \infty$? It is certainly conceivable that $Z(t)$ could converge exponentially to its limiting value, even though the densities $\rho_\omega$ exhibit neutral dynamics as $t \to \infty$. In fact, $Z(t)$ depends only on the order-one Fourier coefficients of $\rho_\omega$, so the higher-order Fourier coefficients could have neutral dynamics (or worse) and have no effect on $Z(t)$. The main result of this paper answers this question: in general, $Z(t)$ does not converge exponentially to its limiting value, even for small perturbations from fixed states in **S**.

The basis for the neutral stability in the Kuramoto model is the Möbius group invariance; the time-$t$ evolution of each density $\rho_\omega$ lies in its $G$-orbit. But, what if one considers the model (2) restricted just to the Poisson densities? Then the argument for neutral stability above no longer applies, and it is conceivable that in this reduced model, the fixed states are indeed asymptotically stable. This in turn would imply that the order parameter $Z(t)$ converges exponentially to its limiting value, for sufficiently small perturbations inside the reduced model of the fixed states. But as we shall see, the order parameter $Z(t)$ need not converge exponentially to its limiting value, even when Eq. (2) is reduced to the invariant subspace of Poisson densities. This is the main result of this paper, which will be proved in detail below. Therefore, the neutral stability for fixed states persists even in the reduced Kuramoto model with Poisson densities.

So is there any way to obtain exponential asymptotic stability for the fixed states, perhaps by restricting the initial states even further? Yes, and this was done by Ott and Antonsen in Ref. 6. They make the additional assumption that the function $\omega \mapsto \zeta_\omega$ which parametrizes the center of mass of the Poisson densities $\rho_\omega$ has a bounded analytic continuation to the upper half plane $\{\operatorname{Im}\omega > 0\}$ and converges to 0 as $\operatorname{Im}\omega \to \infty$. With this assumption, and also assuming that the frequency distribution function satisfies the same condition except for finitely many poles in the upper half plane, they prove that the order parameter $Z(t)$ converges exponentially to a constant. (Note: Ott and Antonsen work with the complex conjugate of the function $\omega \mapsto \zeta_\omega$, so their analytic continuations are into the lower half plane.)

In some sense, this paper is complementary to the important work in Ref. 6. Ott and Antonsen's analytic continuation requirement is actually quite strong; to see this, consider the conformally equivalent setting of the unit disc and its boundary. An $L^2$ function $f$ on the unit circle $S^1$ has an analytic extension to the unit disc if and only if its Fourier coefficients $\hat{f}(n) = 0$ for all $n < 0$. So, the functions which satisfy the analog of Ott and Antonsen's condition for the circle form a very small subspace of $L^2(S^1)$! Ott and Antonsen's condition for functions on the real line is similarly quite restrictive; for example, it fails for a (nonzero) function $f$ if $f$ is even, odd, real-valued or has compact support on $\mathbb{R}$ (in each case, $f$ is entire by the Schwarz reflection principle, and then must be 0 by Liouville's theorem). In fact, as a corollary to our main theorem, we will prove that if the initial parametrization of the Poisson densities $\omega \mapsto \zeta_\omega$ is continuous and vanishes on an interval, then the order parameter $Z(t)$ cannot converge to 0 exponentially fast. So, even for some very nice initial conditions, the order parameter $Z(t)$ does not converge exponentially to its limiting value.

We note for historical completeness that the author and collaborators obtained similar results for the decay of the order parameter twenty years ago for the *linearized* infinite-$N$ Kuramoto model in the subcritical range in Ref. 7, where the decay mechanism was shown to be similar to Landau damping in plasmas. In particular, it was shown that for general initial conditions the order parameter need not converge to 0 exponentially fast, and in fact need not converge to 0 at all! Unfortunately, the results in Ref. 7 do not necessarily extend to the actual *nonlinear* infinite-$N$ Kuramoto model we study, precisely because of its highly neutral stability around the fixed states. This present work can be seen as filling in this gap.

The organization of this paper is as follows: we begin with a detailed description of the reduced Kuramoto model for the Poisson densities, and then for motivation investigate the behavior of $Z(t)$ in the uncoupled case $K = 0$, which turns out to be very instructive. Next, we turn to the case $K > 0$ and prove our main result, which is that if $Z(t) \to 0$ exponentially, then the initial condition must satisfy a regularity condition which we will describe below. We conclude with some ideas for future research on this problem.

## II. REDUCED SYSTEM FOR POISSON DENSITIES

A Poisson density $\rho$ is completely determined by its center of mass

$$\zeta = \int_0^{2\pi} e^{i\theta}\rho(\theta)\, d\theta$$

(see Ref. 5). So, a state for the reduced Kuramoto model is just a complex-valued measurable function $f$ on $\mathbb{R}$ with $|f(\omega)| \le 1$ a.e. which parameterizes the center of mass $\zeta_\omega = f(\omega)$ as a function of the frequency $\omega$. In other words, the state space for the reduced Kuramoto system is just the unit ball in $L^\infty(\mathbb{R})$. Henceforth, we will use the real variable $x$ instead of $\omega$ to parametrize the densities. As shown in Refs. 5 and 6, the Kuramoto system on the subspace of Poisson densities reduces to these equations for the time–$t$ evolution of the function $f$

$$\frac{d}{dt}(f_t(x)) = ixf_t(x) + \frac{K}{2}\left(Z(t) - \overline{Z(t)}\, f_t(x)^2\right),$$
$$\text{where} \quad Z(t) = \int_{-\infty}^{\infty} f_t(x)g(x)\, dx. \tag{3}$$

It is not hard to show that if $||f_0||_\infty \le 1$ then the evolution $f_t$ under (3) satisfies $||f_t||_\infty \le 1$ for all $t \ge 0$, so the time- $t$ flow exists for all $t > 0$.

We make the usual assumptions on the density function: $g$ is positive and continuous on $\mathbb{R}$, $g(-\omega) = g(\omega)$ and $g$ is non-increasing on $[0, \infty)$. We will also need a mild analyticity condition on $g$: We assume that $g$ has an analytic continuation to a nonzero function on some horizontal strip $0 < \operatorname{Im} z < c$. This is true for the standard choices for $g$, namely, Gaussian or Lorentzian densities. We wish to address whether the following condition holds for the asymptotic behavior of the order parameter $Z$:

*Exponential convergence condition*: $Z(\infty) = \lim_{t\to\infty} Z(t)$ exists, and for some constants $c, C > 0$,

$$|Z(t) - Z(\infty)| \le Ce^{-ct}$$

for all $t > 0$.

As we shall see, the truth of this condition depends strongly on the initial condition $f_0$. If, for example, the initial condition is a fixed state for the system, then $Z(t)$ is constant so this condition holds trivially. More importantly, it holds for the special initial conditions studied in Ref. 6. Note that in the subcritical case $K < K_c$, one would expect that $Z(\infty) = 0$, so the condition is just $Z(t) = O(e^{-ct})$ for some $c > 0$; this is the case we shall subsequently focus on.

## III. THE CASE $K = 0$

We can gain a lot of insight into what is happening from the special case $K = 0$. Then, the system uncouples and we have

$$f_t(x) = e^{ixt} f_0(x),$$

$$Z(t) = \int_{\infty}^{\infty} e^{ixt} f_0(x) g(x)\, dx.$$

So, $Z(t) = \sqrt{2\pi}\,\widehat{f_0 g}(-t)$, where $\hat{}$ denotes Fourier transform (following the convention in Ref. 8 (p. 178)], we use the prefactor $1/\sqrt{2\pi}$ in defining the Fourier transform). The function $f_0 g$ is $L^1$, $L^\infty$ and hence also $L^2$ on $\mathbb{R}$; therefore, $\widehat{f_0 g}$ is continuous and bounded on $\mathbb{R}$, and $\lim_{t\to\infty} Z(t) = 0$ (Riemann–Lebesgue lemma for $L^1$ functions). Now, suppose the exponential convergence condition holds: there exists $c > 0$ such that $Z(t) = O(e^{-ct})$ for $t > 0$; i.e., $\widehat{f_0 g}(t) = O(e^{ct})$ for $t < 0$. The following somewhat technical lemma shows that this has strong consequences for the function $f_0 g$. The results in the lemma will be familiar to experts in Fourier transform theory; but since they are not easy to find in precisely the form we will need, we include them for the sake of completeness. The proof is deferred to an appendix so as not to interrupt the arguments leading to the main result of this paper. And to keep the notation as simple as possible, we state the lemma in terms of an arbitrary $L^2$ function $f$; later, we will apply it to the function $f_0 g$ discussed above.

*Lemma 3.1*. Let $f \in L^2(\mathbb{R})$ and suppose there exists $c > 0$ such that $\hat{f}(t) = O(e^{ct})$ for $t < 0$.

1. *The function F defined by*

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itz} \hat{f}(t)\, dt$$

*is holomorphic on the region* $0 < \Im z < c$.

2. *If f is bounded on* $\mathbb{R}$, *then F is bounded on any strip* $0 < \Im z \le c' < c$.
3. *We have* $\lim_{y\to 0^+} F(x+iy) = f(x)$ *for almost all* $x \in \mathbb{R}$.
4. *If f is continuous at some* $x \in \mathbb{R}$ *then F has boundary value f(x) at x.*
5. *If f vanishes on some open interval in* $\mathbb{R}$, *then f (and therefore F) is identically* 0 *a.e.*

So, if the order parameter $Z(t)$ decays exponentially to 0 for the $K = 0$ reduced system with initial condition $f_0$, this lemma implies that the function $f_0 g$ admits an analytic extension to the strip $0 < y < c$, in the sense described in the lemma. We assumed that the density function $g$ has a continuous, nonzero analytic extension to a strip $0 \le y < c'$ for some $c'$; hence, the initial condition $f_0 = f_0 g/g$ also has an analytic extension to the strip $0 < y < \min(c, c')$. In other words, exponential decay of the order parameter can only occur if the initial condition $f_0$ admits an analytic extension into a strip in the upper half plane. By Eq. (5) in the lemma, this condition fails, for example, if a (not identically 0) $f_0$ vanishes on an interval, so for such initial conditions $Z(t)$ does not decay exponentially. In the next section, we will see that a similar result holds when we turn on the coupling $K > 0$.

## IV. THE CASE $K > 0$

We now prove the main result of this paper, which extends the result for $K = 0$ above. To keep the technical details to a minimum, we henceforth assume that the initial condition $f_0$ is continuous on $\mathbb{R}$, which implies that $f_t$ is continuous for all $t > 0$. Our result is relevant mainly to the subcritical case $0 < K < K_c$, where Kuramoto's prediction is that the order parameter $Z(t) \to 0$, although we believe a similar result holds for $K > K_c$ as well.

**Theorem 4.1.** *Suppose the order parameter Z(t) for the solution to Eq. (3) with continuous initial condition $f_0$ decays exponentially to $Z(\infty) = 0$. Then, $f_0$ has an analytic extension to an open set U in the upper half plane with $\partial U = \mathbb{R}$.*

*Corollary 4.2. If in addition $f_0$ vanishes on some interval in $\mathbb{R}$, then $f_0 = 0$.*

*Proof of Corollary*. By the Schwartz reflection principle, $f_0$ can be continued analytically into the lower half plane across the interval on which $f_0$ vanishes. But then $f_0$ vanishes on an interval in its domain, and hence is identically zero. □

*Proof of Theorem*. The assumption is that $|Z(t)| \le Ce^{-ct}$ for some constants $c, C > 0$. Since $Z(t)$ is decaying exponentially, we would expect that as $t \to \infty$, $f_t(x) \approx e^{ixt} h_\infty(x)$ for some function $h_\infty$. To derive this precisely, let

$$h_t(x) = e^{-ixt} f_t(x);$$

then $h_t$ satisfies the equation

$$\frac{d}{dt}(h_t(x)) = \frac{K}{2}\left(e^{-ixt} Z(t) - e^{ixt} \bar{Z}(t) h_t(x)^2\right). \tag{4}$$

The RHS is bounded by $CKe^{-ct}$, so for any $0 < t_1 < t_2$ we have

$$|h_{t_2}(x) - h_{t_1}(x)| \le CK \int_{t_1}^{t_2} e^{-ct}\, dt = \frac{CK}{c}\left(e^{-ct_1} - e^{-ct_2}\right).$$

This implies that $h_\infty(x) = \lim_{t\to\infty} h_t(x)$ exists for all $x \in \mathbb{R}$ and

$$|h_t(x) - h_\infty(x)| \le \frac{CK}{c} e^{-ct},$$

so the convergence is uniform in $x$ and hence $h_\infty$ is continuous on $\mathbb{R}$.

Now

$$Z(t) = \int_{-\infty}^{\infty} e^{ixt} h_t(x) g(x)\, dx,$$

so the Fourier transform of the function $h_\infty g$ for $t < 0$ can be expressed as

$$\begin{aligned}\widehat{h_\infty g}(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixt}(h_{-t}(x) + (h_\infty(x) - h_{-t}(x))g(x)\, dx \\ &= \frac{1}{\sqrt{2\pi}} Z(-t) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixt} \\ &\quad \times (h_\infty(x) - h_{-t}(x))g(x)\, dx,\end{aligned}$$

and this gives the estimate

$$|\widehat{h_\infty g}(t)| \le \frac{C}{\sqrt{2\pi}}\left(1 + \frac{K}{c}\right) e^{ct}$$

for $t < 0$. Hence, $h_\infty g$ satisfies the hypothesis of our lemma, and so admits an analytic extension to the strip $0 < y < c$. Therefore, we see that the function $h_\infty = (h_\infty g)/g$ also extends analytically to some strip $0 < y < \tilde{c}$, with $0 < \tilde{c} \le c$.

The idea now is to work backwards from $h_\infty$ to $h_0 = f_0$ and show that this function also admits an analytic extension into the upper half plane. Consider the complex differential equation which extends (4)

$$\frac{dw}{dt} = \frac{K}{2}(e^{-izt} Z(t) - e^{izt} \bar{Z}(t) w^2). \tag{5}$$

This is a Riccati equation in $w$; the coefficients depend continuously on $t$ and analytically on the parameter $z \in \mathbb{C}$. The general Riccati equation has the form

$$\frac{dw}{dt} = a(t)w^2 + b(t)w + c(t), \tag{6}$$

where we assume that the coefficients are continuous complex-valued functions in $t$. The beauty of these equations is that they extend to the Riemann sphere $\hat{\mathbb{C}}$; to see this, let $v = 1/w$ and observe that Eq. (6) transforms to

$$\frac{dv}{dt} = -c(t)v^2 - b(t)v - a(t),$$

which is also a Riccati equation. Since $\hat{\mathbb{C}}$ is compact, for any initial condition the solution to Eq. (6) exists for all time $t$.

Let $\phi_{t,z}(w)$ be the time-$t$ flow for Eq. (5) with initial condition $w$ and parameter $z$; this flow exists for all $t \in \mathbb{R}$, $z \in \mathbb{C}$, and $w \in \hat{\mathbb{C}}$. For fixed $t$ and $z$, the map $\phi_{t,z}$ is a holomorphic invertible map on $\hat{\mathbb{C}}$, and hence is a Möbius transformation. Furthermore, for any $t$ the map

$$\Phi_t : \mathbb{C} \times \hat{\mathbb{C}} \to \mathbb{C} \times \hat{\mathbb{C}}$$

defined by

$$\Phi_t(z, w) = (z, \phi_{t,z}(w))$$

is a holomorphic invertible map on $\mathbb{C} \times \hat{\mathbb{C}}$ because the coefficients in Eq. (5) depend analytically on $z$.

Now, consider Eq. (5) restricted to $z = x + iy$ with $|y| \le c' < c$ for some $c'$. Then, for $t \ge 0$,

$$|e^{\pm izt}| = e^{\mp yt} \le e^{|y|t}$$

so the coefficients in Eq. (5) are dominated by $e^{(|y|-c)t} \le e^{(c'-c)t}$, uniformly in $z$ (note that this also holds near the point at infinity on $\hat{\mathbb{C}}$ with respect to the coordinate $v = 1/w$). This implies that for any $z$ in this strip, the flow map $\phi_{t,z}$ converges uniformly on $\hat{\mathbb{C}}$ to a holomorphic map of $\hat{\mathbb{C}}$ to itself, which must be invertible and hence is a Möbius map $\phi_{\infty,z}$; furthermore, the convergence is uniform in $z$. (The distance between maps $\phi_1, \phi_2$ on $\hat{\mathbb{C}}$ is defined as

$$d(\phi_1, \phi_2) = \max d_{\hat{\mathbb{C}}}(\phi_1(w), \phi_2(w)),$$

where $d_{\hat{\mathbb{C}}}$ is the metric on the Riemann sphere.) Hence, the map $\Phi_\infty$ defined on $\{|y| \le c'\} \times \hat{\mathbb{C}}$ by

$$\Phi_\infty(z, w) = \lim_{t\to\infty}(z, \phi_{t,z}(w)) = (z, \phi_{\infty,z}(w))$$

is a uniform limit of holomorphic maps, and hence $\Phi_\infty$ is holomorphic and invertible on the domain $\{|y| < c\} \times \hat{\mathbb{C}}$. The relation between the functions $f_0$ and $h_\infty$ is

$$h_\infty(x) = \phi_{\infty,x}(h_0(x)) = \phi_{\infty,x}(f_0(x))$$

for $x \in \mathbb{R}$, so

$$f_0(x) = \phi_{\infty,x}^{-1}(h_\infty(x)).$$

For real $z$, we can express $f_0(z)$ as the composition of maps

$$z \mapsto (z, h_0(z)) \mapsto \Phi_\infty^{-1}(z, h_0(z)) = (z, f_0(z)) \mapsto f_0(z).$$

Since $\Phi_\infty^{-1}$ is a holomorphic map, this shows that $f_0$ extends to a holomorphic map from the strip $0 < y < \tilde{c}$ (the domain of $h_0$) to $\hat{\mathbb{C}}$; in other words, $f_0$ extends to a meromorphic function on the strip $0 < y < \tilde{c}$. (This argument also shows that $f_0$ is continuous on $\mathbb{R}$ if and only if $h_\infty$ is continuous.) We cannot conclude that $f_0$ is holomorphic on the strip $0 < y < \tilde{c}$, since $\phi_{\infty,z}^{-1}$ may have a pole at some value $h_\infty(z)$. But, since we assumed $f_0$ is continuous, we can conclude that its extension to $0 < y < \tilde{c}$ must be holomorphic in some open set $U$ in the upper half plane with $\partial U = \mathbb{R}$, which completes the proof. □

## V. DISCUSSION

In this paper, we studied the behavior of the Kuramoto order parameter $Z(t)$. In the subcritical case, the expectation based on Kuramoto's predictions is that $Z(t)$ should converge to zero exponentially. We showed that this is only true if the initial condition satisfies an analyticity condition, which does not hold generically. We believe that a similar result also holds if the order parameter converges to some $Z(\infty) \neq 0$, but the connection to Fourier transforms in this case is less direct, so there may be some technical complications to surmount in this case. We hope to address this problem in future research. The main conclusion that can be drawn from our result is that the long-term collective dynamics of the infinite-$N$ Kuramoto model are, to say the least, very subtle. The work of Ott and Antonsen is quite remarkable in this light; they found a special set of initial conditions which reduce the order parameter to a finite-dimensional dynamical system with exponentially attracting fixed points. Our work is complementary to their result, in that we proved that their results do not extend to more general initial conditions.

There are still many mysteries to be solved for the Kuramoto model. For example, it has never been proved that for general initial conditions, the order parameter $Z(t)$ even has a limit as $t \to \infty$! One might suspect this is true, since in the case $K=0$ it holds at least for the reduced system with Poisson densities, by the Riemann–Lebesgue lemma. Another important problem is to understand the extent to which the Ott–Antonsen ansatz captures the full dynamical behavior of the Kuramoto model, at least qualitatively (this point is carefully discussed in Ref. 9). There has been some progress in this direction: in Refs. 10 and 11, it is shown that, with certain analyticity assumptions on both the density function $g$ and initial density $\rho_\omega$, the long-term evolution of the order parameter $Z(t)$ is asymptotically identical to the case for initial conditions in the subspace of Poisson densities, satisfying similar analyticity conditions. What remains mysterious (at least to this author) is to what extent these analyticity conditions are natural, and more importantly, how are they related to the dynamics of the large but finite-$N$ Kuramoto model? Indeed, perhaps the ultimate holy grail in all of this is to understand the relation of the infinite-$N$ Kuramoto model (or its various reductions) to finite-$N$ Kuramoto systems; in other words, what does the analysis of the infinite-$N$ system tell us about the long-term behavior of the more realistic finite-$N$ models? Even after decades of study, there are still very few rigorous results in this direction, and Kuramoto's ingenious heuristic calculations remain as tantalizingly out of reach as when first seen almost 40 years ago.

## APPENDIX: PROOF OF LEMMA

*Proof.* Let us first treat the special case where $\hat{f}(t)=0$ for $t<0$. Then, as is shown in Ref. 8 (p. 371), F(z) is holomorphic in the upper half plane, which gives Eq. (1). Fix $z = x+iy$ with $y>0$ and let $\phi_z(t)=e^{izt}$ for $t>0$, $\phi_z(t)=0$ otherwise. We have $|\phi_z(t)| = e^{-yt}$ for $t>0$ so $\phi_z \in L^1 \cap L^2$ on $\mathbb{R}$. An easy calculation shows

$$\hat{\phi}_z(s) = \frac{1}{i\sqrt{2\pi}} \cdot \frac{1}{s-z}.$$

Using Parseval's identity, we get

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_z(t)\hat{f}(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{i\sqrt{2\pi}} \cdot \frac{1}{s-z} f(s)\,ds$$
$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(s)}{s-z} ds.$$

In the same vein, let $\psi_z(t) = e^{i\bar{z}t}$ for $t<0$, $\psi_z(t)=0$ otherwise. Then

$$\hat{\psi}_z(s) = -\frac{1}{i\sqrt{2\pi}} \cdot \frac{1}{s-\bar{z}}$$

and Parseval's identity gives

$$0 = \int_{-\infty}^{\infty} \psi_z(t)\hat{f}(t)\,dt = \int_{-\infty}^{\infty} \frac{f(s)}{s-\bar{z}} ds.$$

Hence

$$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{1}{s-z} - \frac{1}{s-\bar{z}}\right) f(s)\,ds$$

or equivalently

$$F(x+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(t-x)^2+y^2} f(t)\,dt,$$

which is the Poisson integral of $f$ for the upper half plane. If $f$ is bounded on $\mathbb{R}$ then

$$|F(x+iy)| \le ||f||_\infty \cdot \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(t-x)^2+y^2} dt = ||f||_\infty$$

for any $x+iy$ with $y>0$, which proves Eq. (2). We know from the theory of the Poisson integral that

$$\lim_{y\to 0^+} F(x+iy) = f(x)$$

for almost all $x \in \mathbb{R}$, and that $F$ has boundary value $f(x)$ if $f$ is continuous at $x \in \mathbb{R}$; this gives Eqs. (3) and (4).

Next, we prove Eqs. (1)–(4) for the general case. We express the $L^2$ function $f = f_+ + f_-$, where $\hat{f}_+(t)=0$ for $t<0$ and $\hat{f}_-(t)=0$ for $t>0$. We also express $F = F_+ + F_-$, where

$$F_+(z) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{itz}\hat{f}(t)\,dt \quad \text{and}$$
$$F_-(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{itz}\hat{f}(t)\,dt.$$

For $t<0$, we have $|\hat{f}(t)| \le Ce^{ct}$ for some $C>0$, so the integrand in $F_-(x+iy)$ is dominated by $Ce^{(c-y)t}$. By the same argument as above, $F_-$ is defined and holomorphic in the half plane $y<c$, and therefore $F$ is holomorphic in the strip $0<y<c$ which gives Eq. (1) in the general case. Furthermore $\hat{f}_-$ is $L^1$, so by Fourier inversion $f_-(x) = F_-(x)$ a.e. [Ref. 8 (p. 187)]. Therefore, we can assume $f_-$ is continuous on $\mathbb{R}$, and using Eq. (2) for the special case applied to $f_+$, we see that

$$\lim_{y\to 0^+} F(x+iy) = \lim_{y\to 0^+} (F_+(x+iy) + F_-(x+iy)) = f_+(x) + f_-(x) = f(x)$$

for almost all $x \in \mathbb{R}$, proving Eq. (3), in general. We also have

$$|F_-(x+iy)| \le \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(c-y)t}\, dt = \frac{C}{\sqrt{2\pi}} \frac{1}{c-y},$$

which shows that $F_-$ is bounded on any half plane $y \le c' < c$, and, in particular, $f_-$ is always bounded. Conclusions (2) and (4) of the lemma now follow from the special case above, since $f_-$ is bounded and continuous on $\mathbb{R}$ and $F_-$ is bounded and continuous on $y \le c' < c$.

Finally, for Eq. (5) observe that if $f$ vanishes on an open interval $I \subset \mathbb{R}$, then $F$ extends continuously to $I$ and then can be extended as a holomorphic function across $I$ into the lower half plane by the Schwarz reflection principle. But then $F$ extends to a holomorphic function which vanishes on an interval, and hence $F$ must be identically 0; then Eq. (3) implies that $f = 0$ a.e. on $\mathbb{R}$. □

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