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LETTER TO THE EDITOR

Collective synchronisation in lattices of non-linear oscillators with randomness

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Abstract. We study mutual synchronisation in a model of interacting limit cycle oscillators with random intrinsic frequencies. It is shown rigorously that the model exhibits no long-range order in one dimension, and that in higher-dimensional lattices, large clusters of synchronised oscillators necessarily have a sponge-like structure. Surprisingly, the phase-locking behaviour of the mean-field model is completely different from that of any finite-dimensional lattice, indicating that $d=\infty$ is the upper critical dimension for phase locking.

Large populations of interacting non-linear oscillators can spontaneously synchronise themselves to a common frequency, even if there is some distribution of natural frequencies across the population [1, 2]. This remarkable collective phenomenon, known as self-synchronisation, is important in lasers [3], Josephson junction arrays [4-6], oscillating chemical reactions [1, 2, 7], and networks of biological oscillators, such as heart pacemaker cells [1], swarms of flashing fireflies [8], and groups of women whose menstrual periods become mutually synchronised [9].

There has been a great deal of recent theoretical work on self-synchronisation [10-20], because it is a model problem for physicists interested in cooperative non-linear dynamics in many-body systems with randomness. Several previous analyses [10-15] have relied on mean-field approximations in which the interactions are assumed to have infinite range.

In this letter we present a novel analysis of synchronisation in a model of *locally* interacting oscillators. In this model [16], each oscillator has only one degree of freedom, its phase [1, 2, 19].

The equations of motion are:

$$\dot{\theta}_i=\omega_i+K\sum_j \sin(\theta_j-\theta_i) \qquad i=1,\ldots,N\equiv L^d. \tag{1}$$

Here θ_i is the phase of the oscillator located at site i in a d-dimensional hypercubic lattice, ω_i is its intrinsic frequency, $K\geqslant 0$ is the interaction strength and the sum extends over the nearest neighbours of site i. The intrinsic frequencies ω_i are assumed to be randomly distributed across the population of oscillators, with normalised number density $\rho(\omega)$. By going into a rotating frame and rescaling (1), $\rho(\omega)$ may be assumed to have zero mean and unit variance. The sinusoidal coupling terms tend to align the phases of interacting oscillators and are therefore analogous to ferromagnetic interactions.

We outline below (details will be presented elsewhere [18]) a proof that the model (1) exhibits no long-range order in one dimension, and that in lattices of higher dimensions, clusters of synchronised oscillators necessarily have a sponge-like structure reminiscent of Ising [21] and percolation [22] clusters. These analogies to equilibrium systems are non-trivial since the lattice studied here is composed of self-sustained non-linear oscillators, each of which is itself a dynamical system far from equilibrium. An unexpected difference from equilibrium systems is that the lattice and mean-field models have completely different phase-locking behaviour, for *any* lattice dimensionality, indicating that the 'upper critical dimension' is infinite.

The oscillator lattice (1) is related to more familiar physical systems, such as the XY classical spin model and arrays of Josephson junctions. If there were no randomness in the ω_i, ($\omega_i \equiv \omega$ for all i), then (1) actually is the XY model at zero temperature, if we identify the variable $\phi_i = \theta_i - \omega t$ with the planar spin at site i. This zero-temperature model has a perfectly ordered solution ϕ_i = constant for all i. In the oscillator model (1) this would represent perfect in-phase synchrony. Disorder may be introduced in several ways. In the XY model, thermal disorder may be included by adding delta-correlated white noise to (1). Then the corresponding Fokker-Planck equation gives the correct equilibrium distribution. Alternatively, one may assume randomness in the coupling K [20], as in random-spin systems. In the present model (1), it is the ω_i that are random. This quenched randomness at each site is motivated by the biological interpretation of (1) as a network of pacemaker neurons or heart cells; the random ω_i represent the cell-to-cell variability of intrinsic firing rate found in real populations of spontaneously oscillatory cells [1].

To introduce our results about phase-locking in the oscillator lattice (1), consider the simple case of $N = 2$ oscillators. Then (1) reduces to

$$\dot{\theta}_1 = \omega_1 + K\sin(\theta_2 - \theta_1) \tag{2a}$$

$$\dot{\theta}_2 = \omega_2 + K\sin(\theta_1 - \theta_2). \tag{2b}$$

Let $\phi = \theta_1 - \theta_2$. Then $\dot{\phi} = \omega_1 - \omega_2 - 2K\sin\phi$. A phase-locked solution with $\dot{\phi} = 0$ can exist if and only if

$$|\omega_1 - \omega_2| \leq 2K. \tag{3}$$

Intuitively, locking occurs whenever the frequency difference is small enough relative to the coupling. In this case, there are *two* phase-locked solutions, one of which is approached asymptotically from almost all initial conditions (figure 1). Thus the pair of oscillators eventually settles into a phase-locked state if (3) is satisfied.

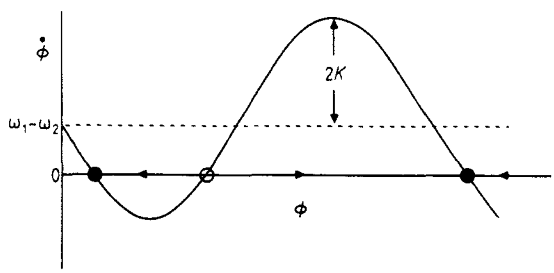

Figure 1. If $|\omega_1 - \omega_2| < 2K$, there are two phase-locked solutions. The stable solution (full circle) is approached asymptotically from almost all initial conditions.

For *random* (ω_1, ω_2), condition (3) may or may not be satisfied and locking may or may not occur. The 'probability of phase locking' $P(K)$ is defined by the probability that (3) is satisfied, i.e.

$$P(K) = \int_\Lambda \rho(\omega_1)\rho(\omega_2)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2$$

where

$$\Lambda = \{(\omega_1, \omega_2): |\omega_1 - \omega_2| \leqslant 2K\}.$$

We now extend the analysis to a chain of N oscillators. The main result is an exact asymptotic ($N \to \infty$) expression for the probability of phase locking. Assume free boundary conditions, i.e. the oscillators at the ends of the chain have only one neighbour. Then (1) becomes

$$\begin{aligned}
\dot\theta_1 &= \omega_1 + K\sin(\theta_2 - \theta_1) \\
\dot\theta_i &= \omega_i + K\sin(\theta_{i+1} - \theta_i) + K\sin(\theta_{i-1} - \theta_i) \qquad 1 < i < N \qquad (4)\\
\dot\theta_N &= \omega_N + K\sin(\theta_{N-1} - \theta_N).
\end{aligned}$$

To calculate the probability of phase-locking, now denoted $P(N, K)$, assume (4) has a phase-locked solution. Then $\dot\theta_i(t) = \dot\theta_j(t)$ for all i, j. Adding all N equations in (4) yields

$$\dot\theta_i(t) = \bar\omega \equiv \frac{1}{N}\sum_{k=1}^{N} \omega_k.$$

Substituting this result and adding only the first j equations gives

$$j\bar\omega = \sum_{i=1}^{j} \omega_i + K\sin(\theta_{j+1} - \theta_j).$$

Equivalently,

$$K\sin\phi_j = X_j \qquad j = 1, \ldots, N-1 \qquad (5)$$

where $\phi_j = \theta_j - \theta_{j+1}$ and $X_j = \Sigma_{i=1}^{j}(\omega_i - \bar\omega)$. Thus a necessary and sufficient condition for a phase-locked solution is that

$$\max_{1 \leqslant j \leqslant N} |X_j| \leqslant K. \qquad (6)$$

If (6) is satisfied, there are in general 2^{N-1} phase-locked solutions of (4) since each of the $N-1$ equations (5) are satisfied by two possible ϕ_j. Precisely one of the locked solutions is approached asymptotically for almost all initial conditions [18, 19]. Hence stable phase-locking occurs whenever (6) is satisfied, and so

$$P(N, K) = \mathrm{Prob}\left(\max_{1 \leqslant j \leqslant n} |X_j| \leqslant K\right). \qquad (7)$$

One expects $\max|X_j| \sim \mathrm{O}(N^{1/2})$ since X_j is a sum of $\sim N$ random variables. This suggests that the coupling needed for phase locking scales as $\mathrm{O}(N^{1/2})$. Indeed we find [18] that

$$\lim_{N\to\infty} P(N, KN^{1/2}) = \frac{(2\pi)^{1/2}}{K}\sum_{j=0}^{\infty} \exp\left(\frac{-(2j+1)^2\pi^2}{8K^2}\right). \qquad (8)$$

The result (8) follows from the observation that the random variable X_j is a discretisation of pinned Brownian motion [18, 23]. In particular, for any *fixed* coupling K, $\lim_{N\to\infty} P(N, K) = 0$ so phase locking is impossible as $N \to \infty$.

Computer simulation of the chain of oscillators (4) reveals a distinctive kind of local cluster formation in the regime of small K where phase locking fails. We define the average frequency of the oscillator at site j by

$$\tilde{\omega}_j = \lim_{t\to\infty}[\theta_j(t) - \theta_j(0)]/t$$

and we define a 'cluster' to be a connected group of oscillators with the same average frequency. Figure 2 shows the average frequency along the chain for different coupling strengths K. Note the formation of longer clusters as K is increased. Figure 2(d) shows that when K is just below the threshold for phase locking, a single break occurs at that oscillator for which $|X_j|$ attains its maximum (figure 2(e)), as predicted by (6). Note that $|X_j|$ is a measure of the *accumulated* randomness along the chain; the break in figure 2(d) does *not* occur between the two oscillators with the most disparate frequencies, as one might have guessed.

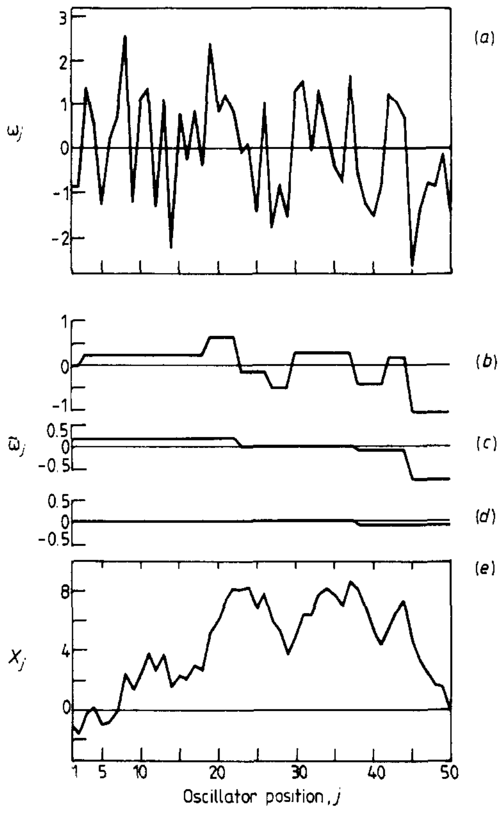

Figure 2. Synchronised clusters in a chain of 50 oscillators. The long-term average frequencies $\tilde{\omega}_j$ (see text) along the chain are shown for: (a) $K = 0$ (these are the intrinsic frequencies ω_j, here sampled from a Gaussian distribution with unit variance); (b) $K = 2$; (c) $K = 4$; (d) $K = 8$. In (e) the critical coupling $K = \max |X_j| \approx 8.46$ occurs at oscillator $j = 37$.

Figure 3 shows the evolution of cluster formation in a 75×75 lattice governed by (1). The grey levels represent the average frequencies at the sites, defined as above but for t finite. White represents the lowest frequency, black the highest. Starting from the initial condition $\theta_j(0)=0$ with random ω_j (figure 3(a)), clusters of synchronised oscillators start to form by $t=10^3$ (figure 3(b)) and become sharply defined by $t=10^4$ (figure 3(c)), which appears to be the asymptotic state for this simulation. Although these simulations suggest that the model (1) gives rise to droplet-shaped clusters, this is an artefact of a finite lattice. In fact, droplet clusters cannot achieve a size of $O(N)$ as $N\to\infty$, as will be shown below.

Generalising now to the case of d-dimensional lattices, we consider the probability that (1) has a solution in which there is a cubical cluster of size $O(N)$. It will be shown that the probability of such 'macroscopic' cubical clusters decays exponentially fast as $N\to\infty$. Hence if solutions to (1) contain any macroscopic clusters, of possibly arbitrary shape, then these clusters must be 'sponge-like' in the sense that they contain no macroscopic cubes.

The idea of the proof is to coarse-grain the system into block oscillators [17, 18]. Consider any cube S containing αN sites. This cube is our putative macroscopic cluster. Divide S into non-overlapping adjacent cubes S_k of size m^d, where $m\sim O(1)$ will be chosen later. These $O(1)$ cubes S_k represent block oscillators. There are $q\approx\alpha N/m^d\sim O(N)$ such blocks, because α, m, and d are fixed independent of N. The frequency Ω_k and phase Θ_k of the block S_k are defined as averages over the sites in S_k:

$$\Omega_k=m^{-d}\sum_{i\in S_k}\omega_i \qquad \Theta_k=m^{-d}\sum_{i\in S_k}\theta_i.$$

Then adding equations (1) for all oscillators $i\in S_k$ and dividing by m^d, we obtain

$$\dot\Theta_k=\Omega_k+Km^{-d}\sum_{i\in S_k}\sum_j\sin(\theta_j-\theta_i). \tag{9}$$

The interaction terms in (9) cancel in pairs, except for the boundary terms with $i\in S_k$ and $j\notin S_k$. There are no more than $2dK/m^{d-1}$ of these terms. Hence for any solution of (1) we have

$$|\dot\Theta_k-\Omega_k|\leqslant 2dK/m. \tag{10}$$

Now suppose in particular that (1) has a solution such that S is a cluster at the frequency λ. Then by time averaging (10) we find that the Ω_k must satisfy the inequalities

$$|\lambda-\Omega_k|\leqslant 2dK/m, \text{ for some } \lambda, \text{ with } k=1,\dots q. \tag{11}$$

It can be shown rigorously [18] that for large but fixed m, the probability that (11) is satisfied is bounded above by $\exp(-cN)$, where $c>0$ is independent of N. This bound depends on rare 'large deviations' of Ω_k, and may be explained heuristically as follows. Inequality (11) is satisfied when $\Omega=(\Omega_1,\dots,\Omega_q)$ lies inside a narrow tube of radius $O(m^{-1})$ about the diagonal line $(\lambda,\lambda,\dots,\lambda)$ in $\boldsymbol{R}^q$. The q-dimensional distribution of Ω is highly concentrated in a ball of radius $O(m^{-d/2})$ about the origin. For $d>2$, this ball lies almost entirely inside the tube where (11) holds. Nevertheless, the distribution has *some* probabilistic mass outside the tube, for sufficiently large but fixed m, because Ω_k can always take on $O(1)$ values with non-zero probability. (In particular, Ω_k has the same $O(1)$ support as the original ω_i.) Because the Ω_k are independent, the probability that (11) is satisfied is $O(\exp(-\text{constant}\times q))\sim O(\exp(-cN))$, for some (very small) $c>0$.

The preceding argument shows that the probability of clustering in any *fixed* cube S of size $\geq \alpha N$ is bounded above by $\exp(-cN)$. Since there are most $\sim O(N)$ possible locations for S in the entire lattice, we conclude that for sufficiently large N,

$$P(N, K, d, \alpha) \leq O(N \exp(-cN)) \to 0 \tag{12}$$

where $P(N, K, d, \alpha)$ denotes the probability that (1) has a solution containing a cubical cluster of αN oscillators.

The inequality (12) has several interesting consequences. First it confirms the conjecture of Sakaguchi *et al* [16] that there is no long-range order in one dimension for the oscillator lattice (1). (Here long-range order means non-zero probability of clusters of size $O(N)$, as $N \to \infty$.) The point is that in one dimension, all clusters are 'cubical' (i.e. segments) and so (12) applies. A similar result has been obtained independently by Daido [17]. Second, for dimensions $d > 1$, although (12) does not rule out $O(N)$-sized clusters of arbitrary shape, it does show that if such clusters exist they contain no large cubes and hence must be riddled with holes. Such sponge-like clusters are reminiscent of those observed in Ising [21] and percolation models [22]. Finally, (12) shows that the probability of phase-locking decays essentially exponentially fast as $N \to \infty$. This is surprising, considering that in the mean-field theory [11], if $\rho(\omega)$ has cutoffs then phase locking occurs with probability one whenever K exceeds a critical coupling K_c. One ordinarily expects mean-field behaviour for lattices of sufficiently large d (above the 'upper critical dimension'). Hence $d = \infty$ is the upper critical dimension for phase locking in the oscillator lattice (1).

An important open problem is whether $O(N)$-sized clusters can be proved to exist, perhaps for d greater than some lower critical dimension. Numerical evidence [16] and heuristic calculations [16–18] suggest that $d = 2$ is the lower critical dimension for clustering in the model (1). If correct, this conjecture would establish another similarity between the model (1) and the XY model, which exhibits no long-range order when $d \leq 2$.

In summary, we have outlined the first rigorous analysis of self-synchronisation in a population of N locally interacting oscillators with random intrinsic frequencies. By moving beyond earlier mean-field approaches, we have found several results that depend on the spatial dimensionality. The one-dimensional lattice studied here requires coupling of size $O(N^{1/2})$ to phase lock; for fixed coupling, it cannot support macroscopic clusters of synchronised oscillators. If such clusters exist in higher-dimensional lattices, they are sponge-shaped, not droplet-shaped. Lastly, the upper critical dimension for phase locking is infinite, indicating that the phase-locking behaviour of the mean-field model and lattice models are completely different.

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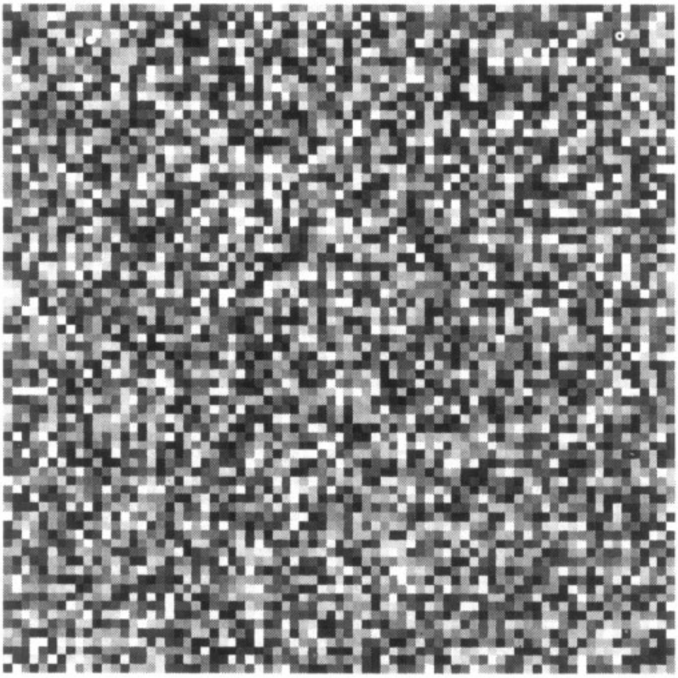
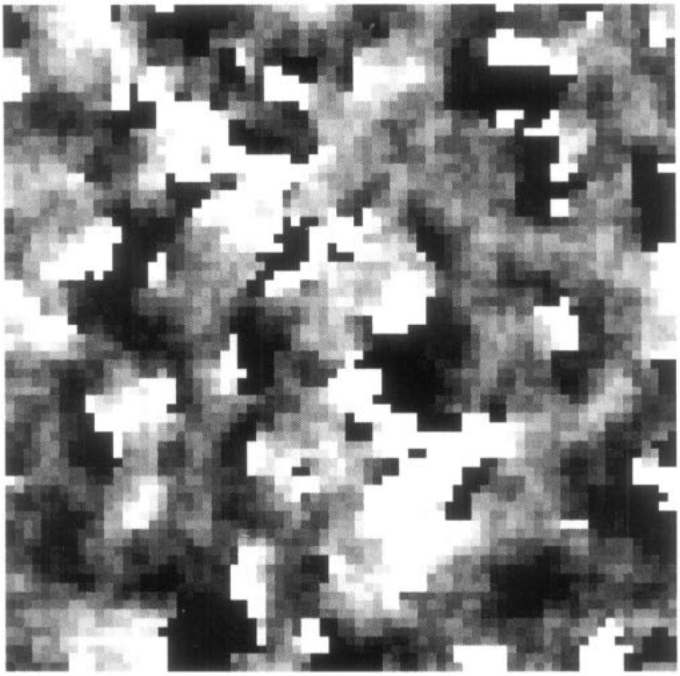


(*a*)

Figure 3. Evolution of locally synchronised clusters in a two-dimensional lattice of oscillators (1) with $K = 0.25$. Grey levels indicate the local frequency, increasing from white to black. (*a*) Intrinsic frequencies ω_j, sampled from a uniform distribution on [0, 1]. (*b*) Coupling-modified frequencies $\tilde{\omega}_j$ at time $t = 10^3$ and (*c*) at $t = 10^4$ when sharply defined clusters have formed.



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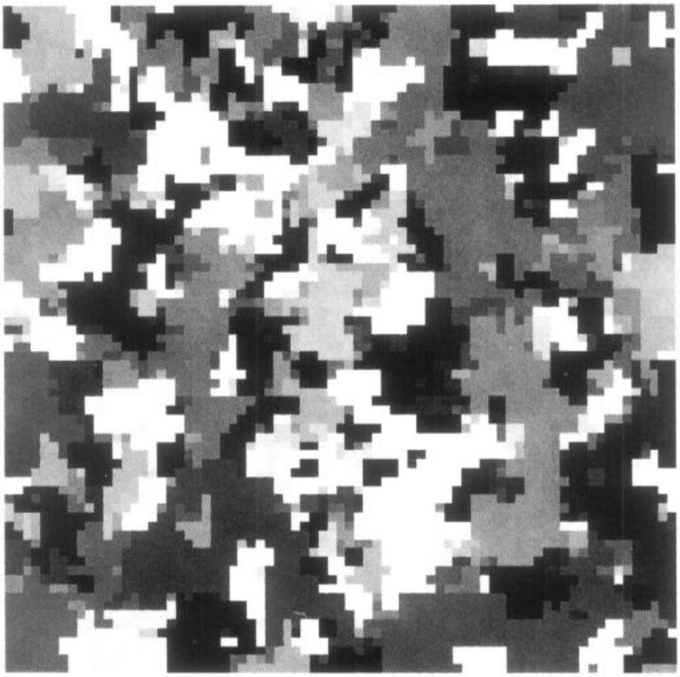

(*c*)

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