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Stability of Incoherence in a Population of Coupled Oscillators

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We analyze a mean-field model of coupled oscillators with randomly distributed frequencies. This system is known to exhibit a transition to collective oscillations: for small coupling, the system is incoherent, with all the oscillators running at their natural frequencies, but when the coupling exceeds a certain threshold, the system spontaneously synchronizes. We obtain the first rigorous stability results for this model by linearizing the Fokker–Planck equation about the incoherent state. An unexpected result is that the system has pathological stability properties: the incoherent state is unstable above threshold, but *neutrally* stable below threshold. We also show that the system is singular in the sense that its stability properties are radically altered by infinitesimal noise.

KEY WORDS: Nonlinear oscillator; synchronization; phase transition; mean-field model; bifurcation; collective phenomena; phase locking.

1. INTRODUCTION

Collective synchronization is a remarkable phenomenon which occurs at practically every level of biological organization.[1,2] Examples range from epileptic seizures in the brain[3] and electrical synchrony among cardiac pacemaker cells[4] to synchronous flashing in swarms of fireflies,[5] the chirping of crickets in unison,[6] and the mutual synchronization of menstrual cycles in groups of women.[7]

Winfree[1] was the first to emphasize the ubiquity of collective synchronization, and also to reduce the problem to its mathematical essence. He modeled each member of the population as a nonlinear oscillator with

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a globally attracting limit cycle. The oscillators were assumed to be weakly coupled and their natural frequencies were assumed to be randomly distributed across the population. Winfree discovered that synchronization occurs *cooperatively*, in a manner strikingly reminiscent of a thermodynamic phase transition. In the absence of coupling, the population behaves incoherently, with each oscillator running at its natural frequency. As the coupling is increased, the population continues to be incoherent until a critical coupling is exceeded—then the system spontaneously "condenses" into a partially synchronized state.

Kuramoto[8,9] provided the next major advance in the study of populations of coupled oscillators. He proposed an analytically tractable model which elucidated the connection between collective synchronization and phase transitions. The governing equation of the model is

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \qquad i = 1,..., N \tag{1.1}$$

Here θ_i is the phase of the ith oscillator, ω_i is its natural frequency, and $K \geqslant 0$ is the coupling strength. The frequencies are chosen at random from a symmetric, one-humped distribution with density $g(\omega)$. By going into a rotating frame if necessary, we may assume that $g(\omega)$ has mean zero.

In the model (1.1), each oscillator is coupled equally to all the others. This is a crucial simplifying assumption, corresponding to the mean-field approximation in statistical mechanics. Kuramoto[8,9] showed that (1.1) could be solved formally in the infinite-N limit, as follows. Consider a complex order parameter defined by

$$re^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \tag{1.2}$$

Here $r(t) \geqslant 0$ measures the phase coherence of the oscillators, and $\psi(t)$ measures the average phase. Then, because of a trigonometric identity, the governing equations may be rewritten as

$$\dot{\theta}_i = \omega_i + Kr \sin(\psi - \theta_i) \tag{1.3}$$

for $i = 1,..., N$. Equation (1.3) shows that the oscillators are coupled only through the mean-field quantities r and ψ. The coupling tends to synchronize the oscillators—each phase θ_i is pulled toward the average phase ψ by a restoring force of strength Kr.

The key insight is that in the infinite-N limit, there are steady solutions in which all fluctuations vanish. In particular, there are self-consistent solutions of (1.3) in which *$r(t)$ and $\psi(t)$ are constant.* For example, one such solution is the completely "*incoherent*" solution with all the oscillators distributed uniformly around the circle and rotating at their natural frequencies. The incoherent solution has $r(t) \equiv 0$ and exists for all values of the coupling strength K. Kuramoto[8,9] showed that a second family of solutions branches off the incoherent solution at a critical coupling strength

$$K_c = \frac{2}{\pi g(0)} \tag{1.4}$$

These new solutions have coherence $r > 0$ and are "partially synchronized," in the sense that the population of oscillators splits into a mutually synchronized group with $|\omega| \leqslant Kr$ and a drifting group with $|\omega| > Kr$. In other words, the oscillators lying near the center of the frequency distribution become locked, whereas the outlying oscillators remain desynchronized. Just above the transition, the coherence r grows continuously as $(K - K_c)^{1/2}$. This result suggests that the onset of synchronization is similar to a second-order phase transition.

Although Kuramoto's analysis[8,9] of the infinite-N limit is elegant and successful in many respects, it fails to address two very important sets of questions. The first set of questions concerns *the differences between infinite and large but finite N*. For any finite number of oscillators, there are inevitable fluctuations in $r(t)$. Daido[10–12] has shown that these fluctuations are typically of size $O(N^{-1/2})$, but they can be amplified close to the onset of synchronization, i.e., for K close to K_c. There have been several recent attempts[10–14] to develop a theory of the fluctuations near the onset of synchronization, but the matter remains controversial.

There is also the possibility of much larger fluctuations, as pointed out by Nancy Kopell (personal communication). For example, in the absence of coupling ($K = 0$), the system (1.1) is known to exhibit Poincaré recurrence,[15] leading to fluctuations of size $O(1)$ for all N. Although these large fluctuations are extremely rare, they are guaranteed to occur eventually. It is unknown whether the coupled system also exhibits Poincaré recurrence. A rigorous treatment of these fluctuations would be a major theoretical contribution.

The second class of open questions concerns the *stability* of the formal solutions obtained by Kuramoto in the infinite-N limit. Kuramoto[9] conjectured that the incoherent solution is stable for $K < K_c$ and unstable for $K > K_c$, but mentioned that, "surprisingly enough, this seemingly obvious fact seems difficult to prove." The first steps have been taken by Kuramoto and Nishikawa,[13,14] who recently presented two different analyses of the stability of the incoherent solution. Both of these analyses involve some approximations whose validity is uncertain. The first analysis[13] is based on an approximate evolution equation for the coherence $r(t)$, while the revised analysis[14] is based on an approximate linear integral equation for the growth of fluctuations in $r(t)$. According to Kuramoto and Nishikawa,[14] the incoherent solution goes from linearly stable to linearly unstable as K increases through K_c.

In this paper we present the first rigorous stability analysis of the incoherent solution for the infinite-N system. One of our main results is an exact formula for the eigenvalue characterizing the growth rate of coherence for $K > K_c$. By finding where this eigenvalue vanishes, we also obtain a new derivation of the critical coupling K_c given by (1.4).

The analysis reveals that the infinite-N system has pathological stability properties: the incoherent solution is unstable for $K > K_c$, but *neutrally* stable for all $K < K_c$. This result is unusual; in an ordinary second-order phase transition, the disordered state is stable on one side of the transition and unstable on the other. Furthermore, from the point of view of dynamical systems theory, one usually expects neutral stability to hold only at a special parameter value, rather than for a whole interval of parameters.

We also find that the infinite-N system is singular with respect to the addition of infinitesimal amounts of noise: for $K < K_c$ and any noise strength greater than zero, the incoherent solution changes from neutrally stable to linearly stable. The singular limit of zero noise is also associated with a breakdown of uniqueness: whereas in the noisy case there is a unique solution with $r(t) \equiv 0$ for all t, in the noise-free case there is an infinite number of such solutions. Thus if we study the problem for small noise, and then look at the limiting behavior as the noise tends to zero, the results are completely different from those obtained for zero noise. This singular behavior is similar to that seen in fluid mechanics in the limit of zero viscosity—in both cases the highest order derivative is lost from the governing (Navier–Stokes or Fokker–Planck) equation.

The paper is organized as follows. In Section 2 we present the governing equations for the system, using the Fokker–Planck formalism. After defining the incoherent solution, we obtain the linear equations governing the growth of small perturbations about the incoherent state. Fourier methods are then used to show that the first harmonic of the perturbation plays a special role; in Section 3 we study the evolution of the fundamental mode, and thereby obtain a formula for the growth rate of the coherence $r(t)$. The results are illustrated with analytical, graphical, and numerical examples. We also stress the pathological stability properties of the noise-free system. Section 4 deals with the evolution of the higher harmonics of the perturbation. These higher harmonics decay exponentially fast in the presence of noise, but can persist forever in the absence of noise.

Finally, in Section 5 we discuss some open problems and compare our approach to the self-consistent method of Kuramoto.[8,9] We point out that our techniques can be used to analyze problems with nonsinusoidal coupling, for which the self-consistent method breaks down.

2. GOVERNING EQUATIONS

The first problem is to decide what we mean by the infinite-N limit of (1.1). One approach is to replace $\theta_i(t)$, $i=1,..., N$, with a function $\theta(t, \omega)$, where ω ranges over the support of $g(\omega)$. We will not follow this seemingly reasonable approach, for two reasons. First, we are interested in adding noise to (1.1); in this case it is more natural to describe the system in terms of a density $\rho(\theta, t, \omega)$, as discussed below. Second, we want to allow for the possibility that two or more oscillators have the same frequency, as in the case when there are delta functions in $g(\omega)$. In this case θ could not possibly be a single-valued function of ω.

Even if $g(\omega)$ does not contain delta functions, it is still more natural to describe the infinite system in terms of a density $\rho(\theta, t, \omega)$ rather than a function $\theta(t, \omega)$. For example, consider what one means by a random initial condition for the large-N system. For $i=1,..., N$, we pick a frequency ω_i at random from $g(\omega)$, and then we pick a random phase θ_i from a uniform distribution on the circle. For simplicity suppose that $g(\omega)$ is the uniform density on the interval $I=[-\gamma, \gamma]$. Then the pair (ω_i, θ_i) is a point on the cylinder $I\times S^1$. For large N, the points are essentially uniformly distributed on the cylinder. Thus, as $N\to\infty$, the density becomes increasingly well behaved, whereas the function $\theta(t, \omega)$ becomes increasingly irregular.

We now discuss the density approach in more detail.

2.1. Densities and the Fokker–Planck Equation

The infinite system should be visualized as follows: for each frequency ω, there is a continuum of oscillators distributed along the circle. Suppose that this distribution is characterized by a density $\rho(\theta, t, \omega)$. Here $\rho(\theta, t, \omega)\, d\theta$ gives the fraction of oscillators of natural frequency ω which lie between θ and $\theta+d\theta$ at time t. Then the appropriate normalization condition is

$$\int_0^{2\pi} \rho(\theta, t, \omega)\, d\theta = 1 \tag{2.1}$$

for all ω and all t. Furthermore, the density ρ is required to be 2π-periodic in θ.

As mentioned above, we are interested in adding noise to (1.1). Consider the following generalization of (1.1), considered previously by Sakaguchi[16]:

$$\dot{\theta}_i = \omega_i + \xi_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \qquad i = 1,..., N \tag{2.2}$$

where the variables $\xi_i(t)$ are independent white noise processes that satisfy

$$\langle \xi_i(t) \rangle = 0 \tag{2.3a}$$

$$\langle \xi_i(s)\, \xi_j(t) \rangle = 2D\delta_{ij}\delta(s-t) \tag{2.3b}$$

In (2.3), the noise strength D is nonnegative and the angular brackets denote an average over realizations of the noise. In the context of biological oscillators, the noise terms can be interpreted as rapid fluctuations in the intrinsic frequency of the oscillators. In the context of thermodynamic systems, the noise terms represent thermal fluctuations at a temperature proportional to the parameter D.

Since (2.2) is a system of coupled Langevin equations, the evolution of $\rho(\theta, t, \omega)$ is governed by the following Fokker–Planck equation[16,17]:

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial \theta^2} - \frac{\partial}{\partial \theta}(\rho v) \tag{2.4}$$

where the velocity $v(\theta, t, \omega)$ is given by

$$v(\theta, t, \omega) = \omega + Kr \sin(\psi - \theta) \tag{2.5}$$

and the order parameter amplitude $r(t)$ and phase $\psi(t)$ are now defined by

$$re^{i\psi} = \int_0^{2\pi} \int_{-\infty}^{\infty} e^{i\theta} \rho(\theta, t, \omega)\, g(\omega)\, d\omega\, d\theta \tag{2.6}$$

Note that the interaction between oscillators of different frequencies occurs solely through the order parameter (2.6).

2.2. Incoherent Solution

Equations (2.4)–(2.6) govern the evolution of the density $\rho(\theta, t, \omega)$. Our goal is to analyze the evolution of $\rho(\theta, t, \omega)$ in the neighborhood of the incoherent solution, defined as follows.

Definition. The incoherent solution of (2.4) is given by

$$\rho_0(\theta, t, \omega) = \frac{1}{2\pi} \tag{2.7}$$

for all θ, t, and ω.

The incoherent solution ρ_0 corresponds to a state in which, for each ω, all the oscillators are uniformly distributed around the circle. It is easy to verify that ρ_0 is a static solution of the Fokker–Planck equation (2.4): substitution of ρ_0 into (2.6) shows that the corresponding $r(t)$ vanishes identically and hence the velocity (2.5) reduces to $v(\theta, t, \omega) = \omega$. In particular, v is independent of θ and therefore

$$\frac{\partial \rho_0}{\partial t} = D \frac{\partial^2 \rho_0}{\partial \theta^2} = \frac{\partial}{\partial \theta}(\rho_0 v) = 0$$

Thus (2.4) is satisfied.

2.3. Linearized System

Now consider the evolution of a small perturbation away from the incoherent state: let

$$\rho(\theta, t, \omega) = \frac{1}{2\pi} + \varepsilon \eta(\theta, t, \omega) \tag{2.8}$$

where $\varepsilon \ll 1$. The normalization condition (2.1) implies that $\eta(\theta, t, \omega)$ satisfies

$$\int_0^{2\pi} \eta(\theta, t, \omega)\, d\theta = 0 \tag{2.9}$$

for all ω and t, and the Fokker–Planck equation (2.4) implies

$$\varepsilon \frac{\partial \eta}{\partial t} = \varepsilon D \frac{\partial^2 \eta}{\partial \theta^2} - \frac{\partial}{\partial \theta}\left[\left(\frac{1}{2\pi} + \varepsilon \eta\right) v\right] \tag{2.10}$$

Consider (2.10) at lowest order in ε. To find the $O(\varepsilon)$ contribution from the bracketed term in (2.10), we observe that $r(t)$ is $O(\varepsilon)$, and hence $v = \omega + O(\varepsilon)$. More specifically, we find

$$r(t) = \varepsilon r_1(t) + O(\varepsilon^2)$$

where

$$r_1 e^{i\psi} = \int_0^{2\pi} \int_{-\infty}^{\infty} e^{i\theta} \eta(\theta, t, \omega)\, g(\omega)\, d\omega\, d\theta \tag{2.11}$$

Thus, (2.5) implies that $\partial v/\partial\theta = -\varepsilon K r_1 \cos(\psi-\theta)$, and so at $O(\varepsilon)$ the evolution equation (2.10) becomes

$$\frac{\partial\eta}{\partial t} = D\frac{\partial^2\eta}{\partial\theta^2} - \omega\frac{\partial\eta}{\partial\theta} + \frac{K}{2\pi} r_1 \cos(\psi-\theta) \tag{2.12}$$

To analyze (2.12) it is convenient to use Fourier methods. Since the function $\eta(\theta, t, \omega)$ is real and 2π-periodic in θ, we seek solutions of (2.12) of the form

$$\eta(\theta, t, \omega) = c(t, \omega)\, e^{i\theta} + c^*(t, \omega)\, e^{-i\theta} + \eta^{\perp}(\theta, t, \omega) \tag{2.13}$$

where the star denotes complex conjugation and $\eta^{\perp}(\theta, t, \omega)$ contains the second and higher harmonics of η. [Note that η automatically has zero mean, by (2.9).]

The particular form of (2.13) is motivated by the observation that the *first* harmonic of η is distinguished. For example, the first harmonic is the only term that contributes to the coherence $r(t)$. Consequently, η makes no contribution to the final term in (2.12) except through $c(t, \omega)$ and its complex conjugate. To see this, note that $r_1\cos(\psi-\theta) = \mathrm{Re}[r_1 e^{i\psi} e^{-i\theta}]$. Now substituting (2.13) into (2.11) yields

$$r_1 e^{i\psi} = 2\pi \int_{-\infty}^{\infty} c^*(t, \omega)\, g(\omega)\, d\omega$$

and so

$$\begin{aligned} r_1\cos(\psi-\theta) &= 2\pi\,\mathrm{Re}\left[\left(\int_{-\infty}^{\infty} c^*(t,\omega)\, g(\omega)\, d\omega\right) e^{-i\theta}\right] \\ &= \pi\left(\int_{-\infty}^{\infty} c(t,\omega)\, g(\omega)\, d\omega\right) e^{i\theta} + \text{c.c.} \end{aligned} \tag{2.14}$$

where c.c. denotes the complex conjugate of the preceding term. Thus the final term in (2.12) depends only on $c(t, \omega)$ and its conjugate, as claimed.

When (2.13) and (2.14) are substituted into (2.12), we obtain two qualitatively different evolution equations, one for the fundamental amplitude $c(t, \omega)$ and another for $\eta^{\perp}(\theta, t, \omega)$. The next section is concerned with the amplitude equation for the fundamental mode $c(t, \omega)$, and Section 4 deals with the evolution of $\eta^{\perp}(\theta, t, \omega)$.

3. EVOLUTION OF THE FUNDAMENTAL MODE

The amplitude equation for $c(t, \omega)$ is obtained by inserting (2.13) and (2.14) into (2.12) and then equating the coefficients of $e^{i\theta}$ on both sides of the resulting equation. We find

$$\frac{\partial c}{\partial t} = -(D + i\omega)c + \frac{K}{2}\int_{-\infty}^{\infty} c(t, v)\, g(v)\, dv \tag{3.1}$$

The linear equation (3.1) has an interesting structure. For any given frequency ω, the evolution of $c(t, \omega)$ depends on *all* the other frequencies through the terms $c(t, v)$ in the integral. However, the dependence is the same for all frequencies, because the integral is independent of ω! This convenient property stems from the mean-field character of the original model (2.2).

Before analyzing the spectrum of (3.1), we make a few remarks. There is no need to write the evolution equation for c^*, since it is just the conjugate of (3.1). Note also that the coherence $r(t)$ is determined at this order by $c(t, \omega)$, via (2.14). In particular, if $c(t, \omega)$ grows exponentially, so does $r(t)$.

3.1. Discrete Spectrum

Equation (3.1) has both a discrete and a continuous spectrum. To find the discrete spectrum, we seek solutions of (3.1) of the form

$$c(t, \omega) = b(\omega)\, e^{\lambda t} \tag{3.2}$$

where the eigenvalue λ is independent of ω. Substituting (3.2) into (3.1) yields

$$\lambda b = -(D + i\omega)b + \frac{K}{2}\int_{-\infty}^{\infty} b(v)\, g(v)\, dv \tag{3.3}$$

This equation is easy to solve, because the integral in (3.3) is just some unknown constant to be determined self-consistently. Thus, let

$$A = \frac{K}{2}\int_{-\infty}^{\infty} b(v)\, g(v)\, dv \tag{3.4}$$

Solving (3.3) for $b(\omega)$, we find

$$b(\omega) = \frac{A}{\lambda + D + i\omega} \tag{3.5}$$

Now we invoke self-consistency: (3.5) must be consistent with (3.4). Substitution of (3.5) into (3.4) yields either $A = 0$ or

$$1 = \frac{K}{2} \int_{-\infty}^{\infty} \frac{g(v)}{\lambda + D + iv}\, dv \tag{3.6}$$

The solution $A = 0$ is not allowed, because (3.2) and (3.5) would then imply that $c(t, \omega) \equiv 0$ for all ω, and this is not considered an eigenfunction. Thus, (3.6) is the equation for the discrete spectrum of the linear system (3.1).

From now on, we suppose that $g(\omega)$ is an even function, i.e., $g(\omega) = g(-\omega)$ for all ω. We also assume that $g(\omega)$ is "nonincreasing" on $[0, \infty)$, in the sense that $g(\omega) \leqslant g(v)$ for all $\omega \geqslant v$. These two properties hold for the Gaussian, Lorentzian, and uniform distributions, as well as many others of practical interest. Under these two assumptions *one can prove that (3.6) has at most one solution for λ, and if such a solution exists, it is necessarily real* (see the proof of Theorem 2 in ref. 18). Then (3.6) becomes

$$1 = \frac{K}{2} \int_{-\infty}^{\infty} \frac{\lambda + D}{(\lambda + D)^2 + v^2}\, g(v)\, dv \tag{3.7}$$

Equation (3.7) is the one of the main results of this paper. It shows how the eigenvalue λ depends on the noise strength D, the coupling strength K, and the frequency density $g(\omega)$. This eigenvalue governs the linear stability of the fundamental mode: when $\lambda > 0$, the fundamental mode is unstable, and the coherence grows like $r(t) \approx r_0 e^{\lambda t}$.

It is very important to recognize that any solution λ of (3.7) must satisfy the inequality

$$\lambda > -D \tag{3.8}$$

since otherwise the right-hand side of (3.7) is $\leqslant 0$. In particular, for the noise-free case ($D = 0$), there can never be any negative eigenvalues! Hence, *the fundamental mode is never linearly stable for $D = 0$.*

On the other hand, the fundamental mode can be stable if $D > 0$. Equation (3.7) allows us to find the critical coupling K_c at which stability is lost. The critical condition is $\lambda = 0$, which implies

$$K_c = 2 \left[\int_{-\infty}^{\infty} \frac{D}{D^2 + v^2}\, g(v)\, dv \right]^{-1} \tag{3.9}$$

The critical coupling given by (3.9) was obtained previously by Sakaguchi[16] via an extension of Kuramoto's[8,9] self-consistency argument.

He first found self-consistent *static* solutions of the Fokker–Planck equation (2.4), and then showed that solutions with coherence $r > 0$ bifurcate from the incoherent solution along the locus defined by (3.9). We stress that, in contrast to the analysis given here, the analysis of Sakaguchi[16] does not provide any information about stability.

Our analysis provides the first proof that for the noise-free case, the incoherent solution goes unstable for $K > K_c = 2/[\pi g(0)]$, as conjectured by Kuramoto.[9] To see this, let $D = 0$ in (3.7) and let $\lambda \to 0^+$. The kernel function $\lambda/(\lambda^2 + v^2)$ becomes more and more sharply peaked about $v = 0$, yet its integral over the real line equals π for all positive λ. Thus the kernel function approaches $\pi\delta(v)$ as $\lambda \to 0^+$, and so the right-hand side of (3.7) tends to $(K/2)\,\pi g(0)$. Hence $\lambda > 0$ for $K > 2/[\pi g(0)]$, as required.

In Section 3.4 these results will be illustrated for particular densities $g(\omega)$ for which the eigenvalue λ can be found explicitly. But first we compute the remaining part of the spectrum of (3.1).

3.2. Continuous Spectrum

The linear operator L associated with the right-hand side of (3.1) is given by

$$Lb = -(D + i\omega) b + \frac{K}{2} \int_{-\infty}^{\infty} b(v)\, g(v)\, dv \tag{3.10}$$

Recall that the continuous spectrum of L is defined as the set of complex numbers λ such that the operator $L - \lambda I$ is not surjective. Thus we are led to consider the equation

$$-(\lambda + D + i\omega) b + \frac{K}{2} \int_{-\infty}^{\infty} b(v)\, g(v)\, dv = f \tag{3.11}$$

for fixed λ and for an arbitrary function $f(\omega)$. If (3.11) can always be solved for $b(\omega)$, then λ is *not* in the continuous spectrum.

Notice that the integral in (3.11) is independent of ω. As in (3.4), we denote this integral by A. If $\lambda + D + i\omega = 0$ for some ω in the support of g, then (3.11) is not solvable in general. [It is solvable only for constant functions $f(\omega) \equiv A$.] Hence the continuous spectrum contains the set

$$\{-D - i\omega \colon \omega \in \text{Support}(g)\} \tag{3.12}$$

In fact, (3.12) is *all* of the continuous spectrum, for suppose that λ is not in (3.12). Then (3.11) is solvable: from (3.11),

$$b(\omega) = \frac{A - f(\omega)}{\lambda + D + i\omega} \tag{3.13}$$

All we have to do now is show that A can be determined self-consistently. Substituting (3.13) into (3.4) yields

$$A\left(1-\frac{K}{2}\int_{-\infty}^{\infty}\frac{g(v)}{\lambda+D+iv}\,dv\right)=-\frac{K}{2}\int_{-\infty}^{\infty}\frac{f(v)\,g(v)}{\lambda+D+iv}\,dv \qquad (3.14)$$

By assumption, λ is not in the discrete spectrum, and so the coefficient of A is nonzero, by (3.6). Thus (3.14) can be solved for A.

Hence the set (3.12) is the continuous spectrum. Note that it lies on the imaginary axis if $D=0$, but in the left half-plane if $D>0$.

3.3. Graphs of the Spectrum

Now we sketch the discrete and continuous spectra for the case of an even, nonincreasing $g(\omega)$ with support $[-\gamma, \gamma]$, where $\gamma>0$. The aim is to give a more concrete picture of the spectrum, and also to highlight the pathological features of the noise-free case $D=0$.

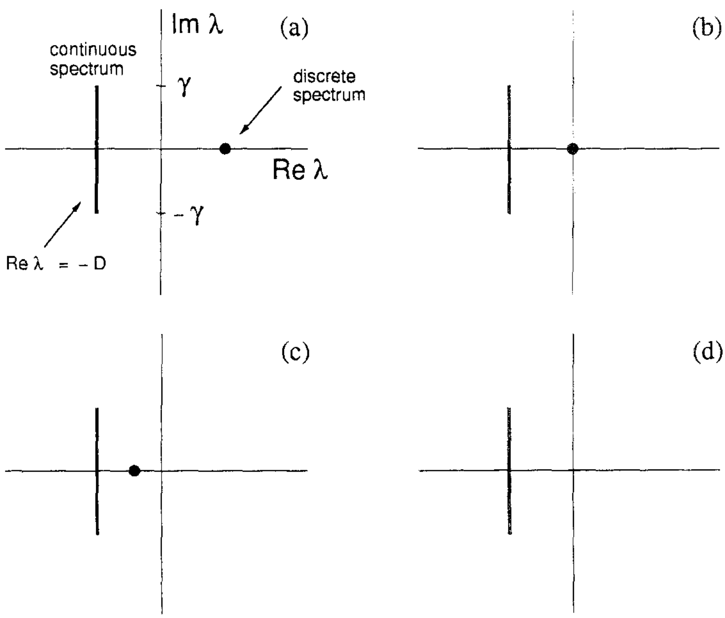

Fig. 1. The continuous and discrete spectra for the linear operator (3.10), for the noisy case $D>0$. (a) $K>K_c$. The fundamental mode is unstable since the discrete spectrum $\lambda>0$. (b) $K=K_c$. The fundamental mode is neutrally stable. (c) $K^*<K<K_c$. The fundamental mode is stable. (d) $K\leqslant K^*$. The discrete spectrum is absorbed by the continuous spectrum and disappears.

First consider the noisy case $D>0$, and suppose that D is fixed. Then the location of the spectrum depends only on the coupling strength K. Figure 1 shows that the discrete spectrum is either a single point (for $K>K^*$, in Figs. 1a–1c) or empty (for $K \leqslant K^*$, in Fig. 1d). Here K^* is defined by the condition $\lambda = -D$, at which the discrete spectrum is born, as indicated by (3.8). The other distinguished value of K is that defined by $\lambda = 0$; as before, we call this value K_c, corresponding to the onset of instability.

Figure 1 shows that for $D>0$, the continuous spectrum (3.12) is a vertical line segment in the left half-plane, irrespective of the value of K. Thus, the modes corresponding to the continuous spectrum never cause instability. In contrast, the discrete spectrum depends strongly on K. When $K>K_c$ (Fig. 1a) the fundamental mode is unstable since $\lambda>0$. As K decreases, the eigenvalue moves to the left—the fundamental mode becomes neutrally stable (Fig. 1b) and then linearly stable (Fig. 1c). Finally, in Fig. 1d, the discrete spectrum is absorbed by the continuous spectrum and disappears.

Figure 2 shows that the pictures are dramatically different when $D=0$. Now the continuous spectrum lies exactly on the imaginary axis. Furthermore, there are only two pictures instead of four as in Fig. 1, because K^* and K_c coincide when $D=0$! Note that the fundamental mode is unstable when $K>K_c$ (Fig. 2a), but *neutrally* stable when $K \leqslant K_c$ (Fig. 2b). As mentioned earlier, the fundamental mode is never linearly stable when $D=0$.

3.4. Exact Solutions for the Eigenvalue

We now study some particular densities $g(\omega)$ for which the eigenvalue determined by (3.7) can be found exactly.

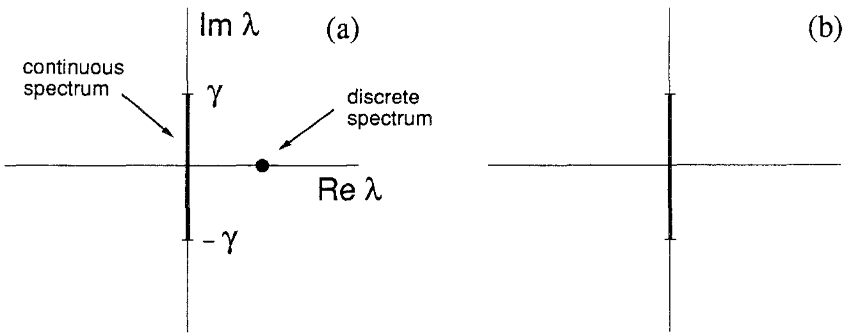

Fig. 2. The continuous and discrete spectra for (3.10), for the noise-free case $D=0$. Compare Fig. 1. (a) $K>K_c$. The fundamental mode is unstable. (b) $K \leqslant K_c$. The fundamental mode is neutrally stable.

(a) Identical Oscillators. Suppose that $g(\omega)=\delta(\omega)$, so that all the oscillators have the same natural frequency. Then the model (2.2) is equivalent to the mean-field theory for a magnetic spin system known as the classical XY model. The noise strength D is interpreted as the temperature and K is the ferromagnetic coupling strength. Equation (3.7) yields

$$\lambda=\frac{K}{2}-D \tag{3.15a}$$

which recovers the well-known result that the incoherent or "paramagnetic" state loses stability at a critical temperature $D_c=K/2$. Below this temperature, the system undergoes spontaneous magnetization (or synchronization, in our context).

(b) Uniform Distribution. Suppose that $g(\omega)=(2\gamma)^{-1}$ for $|\omega|\leqslant\gamma$ and $g(\omega)=0$ otherwise. Then integration of (3.7) yields

$$\lambda= -D+\gamma\cot(2\gamma/K) \tag{3.15b}$$

This result is valid only for $\lambda> -D$, as explained above in (3.8). The same is true for the next two results.

Figure 3 shows a graph of Equation (3.15b). The eigenvalue λ is plotted vs. K, while D and γ are held fixed. To show the relation between λ and the continuous spectrum, we also plot the real part of the continuous spectrum. In Fig. 3a, the discrete spectrum λ emerges from the continuous spectrum at $K=K^*=4\gamma/\pi$. The onset of instability occurs later at $K=K_c=2\gamma/[\tan^{-1}(\gamma/D)]$. Figure 3b shows that once again the case $D=0$ is peculiar; the points K^* and K_c coincide. Thus, the onset of instability occurs exactly when the discrete spectrum is born.

(c) Lorentzian or Cauchy Distribution. If $g(\omega)=(\gamma/\pi)(\gamma^2+\omega^2)^{-1}$, then (3.7) becomes

$$\lambda=\frac{K}{2}-D-\gamma \tag{3.15c}$$

(d) Gaussian Distribution. For a normal distribution with standard deviation σ, we find that λ satisfies the implicit equation

$$1=\left(\frac{\pi}{8}\right)^{1/2}\frac{K}{\sigma}\exp\left(\frac{(\lambda+D)^2}{2\sigma^2}\right)\operatorname{erfc}\left(\frac{\lambda+D}{\sqrt{2}\,\sigma}\right) \tag{3.15d}$$

where erfc denotes the complementary error function.

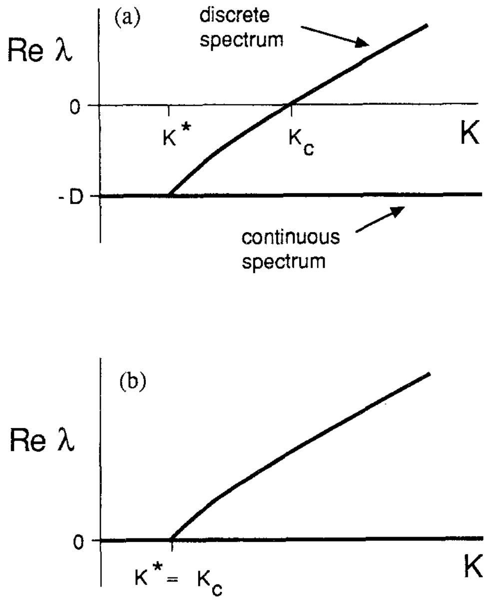

Fig. 3. Graph of (3.15b). The eigenvalue λ is plotted vs. K, for fixed D and γ. The real part of the continuous spectrum is also shown. (a) The noisy case $D>0$. The discrete spectrum λ emerges from the continuous spectrum at $K=K^*$. The onset of instability occurs later at $K=K_c$. (b) For the noise-free case $D=0$, the points K^* and K_c coincide. Hence the onset of instability occurs just as the discrete spectrum is born.

3.5. Numerical Experiments

As a brief check of the analytical results above, we performed the following numerical experiments. These experiments are for the sake of illustration and are certainly not comprehensive.

The system (1.1) was studied for the case of $N=480$ oscillators, with coupling strength $K=1$. The frequencies were uniformly distributed on $[-\gamma, \gamma]$, where $\gamma=0.2$. There was no noise, i.e., $D=0$. The goal was to simulate the evolution of the system starting near the incoherent solution. For these parameters, equation (3.15b) predicts that the coherence $r(t)$ should initially grow exponentially at a rate $\lambda \approx 0.47304$.

Strictly speaking, the incoherent solution (2.7) exists only for infinite N. To approximate the incoherent state for finite N, we chose M evenly spaced frequencies on the interval $[-\gamma, \gamma]$, and assigned N/M oscillators to each frequency. For instance, in the case shown in Fig. 4, the population was partitioned into 40 frequencies, with 12 oscillators at each frequency. Then, for each frequency, the oscillators were evenly spaced around the circle. For a computer with infinite-precision arithmetic, this initial condition would have coherence $r \equiv 0$. Furthermore, $r(t)$ would remain zero for all time. To break the symmetry, we added a random number of size $O(10^{-10})$ to each of the initial phases θ_i, resulting in a nonzero initial coherence.

Figure 4 shows that the coherence initially grows exponentially fast:

$$r(t) \approx r_0 \exp(\lambda_{M,N} t)$$

where $\lambda_{M,N} \approx 0.47164$. The exponential growth of $r(t)$ breaks down when $r \approx 0.1$ because the system is no longer close to the incoherent solution. The measured value of $\lambda_{M,N}$ is within 1% of the λ predicted by the theory for infinite N.

Figure 5 indicates that the deviation of $\lambda_{M,N}$ from λ is a finite-size effect. To demonstrate this, we repeated the simulations for different numbers of frequencies M while holding the total population fixed at $N = 480$ oscillators. Figure 5 shows that

$$\lambda_{M,N} - \lambda \sim O(1/M)$$

as M increases.

This is precisely the dependence on M that one would have expected. By choosing the frequencies to be evenly spaced, we have essentially

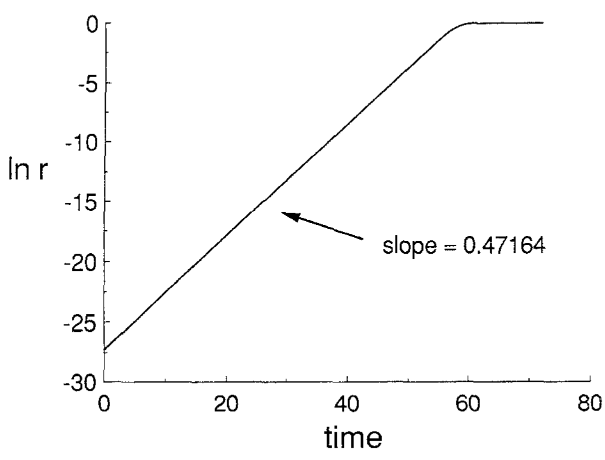

Fig. 4. Linear growth of $\ln r$, corresponding to exponential initial growth of the coherence r, for $K > K_c$. The measured exponential growth rate is $\lambda_{M,N} \approx 0.47164$. Equation (1.1) was integrated numerically using the fourth-order Runge-Kutta method. Parameters: $K = 1$; $N = 480$; uniform $g(\omega)$ with $\gamma = 0.2$; $M = 40$ distinct frequencies; $D = 0$. See text for description of simulation and choice of nearly incoherent initial condition.

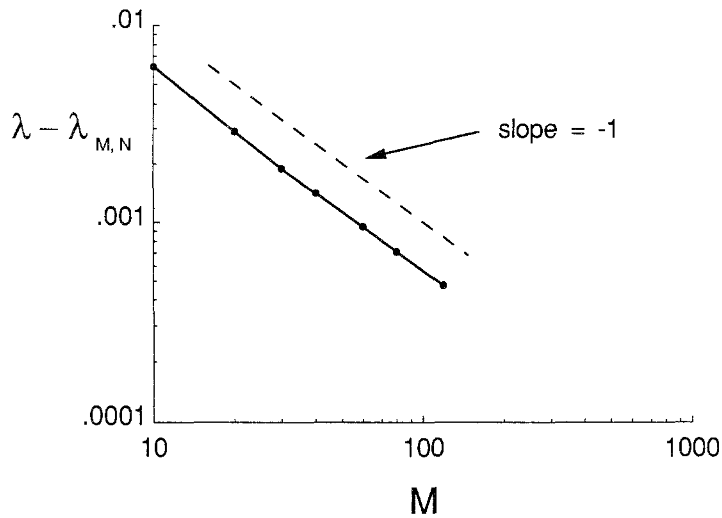

Fig. 5. Log-log plot of $\lambda - \lambda_{M,N}$ as a function of number of distinct frequencies M. Parameters and simulation method as in Fig. 4, except that M varies (M values: 10, 20, 30, 40, 60, 80, 120, shown as solid dots). A dashed line of slope -1 is shown for comparison.

replaced $g(\omega)$ with a comb of M delta functions. This means that the integral in (3.7) is replaced by a Riemann sum—but the deviation between a Riemann sum and its limit is $O(1/M)$. This deviation in the integral in (3.7) leads to a deviation of the same order in λ.

In contrast, the dependence of $\lambda_{M,N}$ on N (with M held fixed) was found to be much weaker. In other words, for the simulation protocol used here, it is more important to sample the frequencies densely than it is to have many oscillators per frequency.

4. EVOLUTION OF HIGHER HARMONICS

Now we complete the linear stability analysis of the incoherent solution. Near the end of Section 2 we derived the evolution equation (2.12) governing the growth of a small perturbation $\eta(\theta, t, \omega)$ about the incoherent solution. It was convenient to express η in terms of two functions, $c(t, \omega)$ and $\eta^{\perp}(\theta, t, \omega)$, as in (2.13). The evolution of $c(t, \omega)$ was discussed in the last section, and is closely related to changes in the coherence $r(t)$; now we study the evolution of the higher harmonics contained in $\eta^{\perp}(\theta, t, \omega)$. Perturbations corresponding to these higher harmonics do *not* lead to changes in $r(t)$.

The main result is that the noise-free case again has pathological stability properties: for $D>0$ the higher harmonics decay exponentially fast, whereas for $D=0$ they persist forever as neutrally-stable rotating waves. Furthermore, there is a breakdown of uniqueness when $D=0$: it turns out that *any* function of the form $\eta^{\perp}(\theta, t, \omega) = f(\theta - \omega t, \omega)$ is a solution of the relevant evolution equation. These rotating wave solutions also occur in the full system, and are not just an artifact of the linearization.

4.1. Linear Stability Analysis

To derive the evolution equation for $\eta^{\perp}(\theta, t, \omega)$, we substitute (2.13) and (2.14) into (2.12) and collect terms involving $\eta^{\perp}$. The result is

$$\frac{\partial \eta^{\perp}}{\partial t} = D \frac{\partial^2 \eta^{\perp}}{\partial \theta^2} - \omega \frac{\partial \eta^{\perp}}{\partial \theta} \tag{4.1}$$

Equation (4.1) is much simpler than the evolution equation (3.1) studied in the last section, because there is *no coupling* between oscillators of different frequencies in (4.1).

We solve (4.1) using Fourier methods. Recall that $\eta^{\perp}$ has zero mean, by (2.9), and zero first harmonic, by (2.13). Hence the Fourier series for $\eta^{\perp}$ starts with the second harmonic:

$$\eta^{\perp}(\theta, t, \omega) = \sum_{|k| \geq 2}^{\infty} a_k(t, \omega)\, e^{ik\theta} \tag{4.2}$$

where $a_k = (a_{-k})^*$, since $\eta^{\perp}$ is real. Substitution of (4.2) into (4.1) yields the amplitude equation

$$\frac{\partial a_k}{\partial t} = (-k^2 D - ik\omega)\, a_k$$

which has the general solution

$$a_k(t, \omega) = a_k(0, \omega)\, e^{(-k^2 D - ik\omega)t}$$

Hence

$$\eta^{\perp}(\theta, t, \omega) = \sum_{|k| \geq 2}^{\infty} a_k(0, \omega)\, e^{-k^2 D t} e^{ik(\theta - \omega t)} \tag{4.3}$$

The solution (4.3) shows that $\eta^{\perp}(\theta, t, \omega)$ decays to zero exponentially fast for any $D > 0$. Because the higher harmonics die out so rapidly, the evolution of the total perturbation η is essentially controlled by its fundamental mode.

In contrast, when $D=0$, Eq. (4.3) shows that $\eta^{\perp}$ is an undamped rotating wave, i.e., a function of $\theta - \omega t$ alone. Of course, this result follows directly from (4.1), since *any* function of the form $\eta^{\perp}(\theta, t, \omega) = f(\theta - \omega t, \omega)$ solves (4.1) when $D=0$. The fact that any function of $\theta - \omega t$ is acceptable means that these rotating waves are *neutrally stable* to perturbations involving only higher harmonics.

4.2. Neutral Stability for the Full System

If $D=0$, neutrally-stable rotating waves occur even in the original nonlinear system (2.4)–(2.6). Moreover, this form of neutral stability holds for all values of K. This neutral stability is therefore completely different and much stronger than that associated with the fundamental mode, which is based on a linear analysis, and which occurs only for $K<K_c$.

We first explain the neutral stability physically, and then mathematically. Suppose that $D=0$ and consider a family of densities $\rho(\theta, t, \omega)$ that satisfies

$$0=\int_0^{2\pi} e^{i\theta}\rho(\theta, t, \omega)\, d\theta \tag{4.4}$$

for all ω, at some fixed time t. Physically, this means that for each frequency ω, the oscillators of that frequency make zero contribution to the order parameter (2.6). Then the coherence $r=0$, and so $v(\theta, t, \omega)=\omega$. In other words, each oscillator moves with instantaneous velocity ω, independent of its position on the circle. Hence for each ω, the associated $\rho(\theta, t, \omega)$ will simply rotate rigidly around the circle at frequency ω. But this means that (4.4) will continue to be satisfied at all later times! [A rotated version of ρ still satisfies (4.4).] Note that the density for each ω is oblivious to all the others—they rotate around and around with no interaction between them. Since the densities are effectively uncoupled, the system is *neutrally stable to perturbations involving only higher harmonics*; changing the shape of one or all of the densities will lead to no restoring forces—as long as no first harmonics are introduced.

To put it more mathematically, fix ω and t and let

$$S=\{\rho(\theta, t, \omega)\text{: (4.4) is satisfied}\} \tag{4.5}$$

Then S is invariant under the flow (2.4), as is easily checked:

$$\begin{aligned}
\frac{\partial}{\partial t}\int_0^{2\pi} e^{i\theta}\rho\, d\theta &= \int_0^{2\pi} e^{i\theta}\frac{\partial \rho}{\partial t}\, d\theta \\
&= -\int_0^{2\pi} e^{i\theta}\frac{\partial}{\partial \theta}(\rho v)\, d\theta \\
&= \int_0^{2\pi} ie^{i\theta}\rho v\, d\theta \\
&= i\omega\int_0^{2\pi} e^{i\theta}\rho\, d\theta \\
&= 0
\end{aligned}$$

In this chain of equations we have used (2.4); integration by parts; $v = \omega$ since $r = 0$ at time t; and finally (4.4).

The invariance of S is important because it implies the type of neutral stability claimed above. The fact that S is invariant means that $v = \omega$ for *all* t. Then the solutions of (2.4) are rotating waves of the form $\rho(\theta - \omega t, \omega)$, where ρ is *arbitrary*. This is the strongest possible form of neutral stability: the dynamics restricted to S are effectively uncoupled.

A noteworthy consequence of these results is that when $D = 0$, there are infinitely many different solutions with $r = 0$ for all time. In contrast, when $D > 0$, all the higher harmonics decay as $t \to \infty$, and so the only solution with $r = 0$ for all time is the incoherent solution (2.7).

5. DISCUSSION

5.1. Open Problems

In this paper we have analyzed the linear stability of the incoherent solution (2.7). One important conclusion is that, in the absence of noise, the incoherent solution is unstable for $K > K_c$, but *neutrally* stable for all $K < K_c$. It would be desirable to extend this linear stability analysis to include higher-order terms. For $K < K_c$, do the nonlinear terms stabilize or destabilize the incoherent solution, or leave it neutrally stable?

A related question concerns the long-term behavior of $r(t)$ for $K < K_c$. Although the incoherent solution is linearly neutrally stable, it is still possible that $r(t) \to 0$ as $t \to \infty$, as predicted by Kuramoto and Nishikawa.[14] To give a simple example (kindly pointed out by Y. Kuramoto), suppose $K = 0$, so that the oscillators are uncoupled. In this case our neutral stability results hold trivially. At the same time, it is true that $r(t) \to 0$ as $t \to \infty$ because of the Riemann–Lebesgue Lemma. Unfortunately, the extension of this simple idea to the case $K > 0$ remains problematical.

Another question concerns the stability of the partially synchronized state[9,13,14] which branches off the incoherent solution at $K = K_c$. Can one show that this solution is locally stable for $K > K_c$? [The partially synchronized state cannot be *globally* stable, because the subspace S defined by (4.5) is invariant for all K.]

As mentioned in the Introduction, there is a whole class of unsolved problems concerning fluctuations in the finite-N system (1.1). Promising starts have been made by Daido[10–12] and by Kuramoto and Nishikawa,[13,14] but many difficult questions remain. For example, Daido[11,12] has shown that the fluctuations are characterized by two different critical exponents on either side of the phase transition at $K = K_c$. Is this peculiar behavior related to the unusual properties of the bifurcation at K_c, in which the incoherent state changes from unstable to neutrally stable?

5.2. Self-Consistent Method vs. Fokker–Planck Method

Finally, we would like to comment on some of the novel features of the analysis used in this paper. We have introduced a method for studying the onset of synchronization in mean-field models of coupled oscillators. The strategy is to linearize the relevant Fokker–Planck equation about the incoherent state, and then analyze the resulting linear stability problem. Until now, the only tool available for analyzing mean-field systems of coupled oscillators has been the self-consistency approach pioneered by Kuramoto.[8,9] That approach has the advantage that it allows one to study both incoherent and partially synchronized solutions in one stroke—in a sense, the self-consistent method is global, whereas our method is local.

On the other hand, the self-consistent method has some important limitations. First, it is concerned only with steady-state behavior and it therefore provides no information about stability. Second, it depends crucially on the sinusoidal form of the coupling in the model (1.1). Because of a convenient trigonometric identity, the order parameter (1.2) appears in the governing equation (1.3)—it is this coincidence which allows the order parameter to be determined self-consistently.

In contrast, our method can be applied to systems with more general coupling than (1.1) or (2.2). For example, consider the system

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} f(\theta_j - \theta_i), \qquad i = 1, \ldots, N \tag{5.1}$$

where $f(\phi)$ is an arbitrary 2π-periodic function. Equation (5.1) shares an important qualitative feature with the sinusoidal model (1.1): both systems are coupled only through *phase differences*. This sort of coupling arises naturally from averaging theory applied to a system of weakly coupled limit cycle oscillators.[9]

Using the notation of Section 2, we see that the appropriate infinite-N limit of (5.1) is

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \theta}(\rho v) \tag{5.2}$$

where the velocity is given by

$$v(\theta, t, \omega) = \omega + K \int_{-\infty}^{\infty} \left[\int_0^{2\pi} f(\phi - \theta)\, \rho(\phi, t, v)\, d\phi \right] g(v)\, dv \tag{5.3}$$

This system will be analyzed in detail elsewhere,[19] so we just sketch some of the results. The system is tractable because the term in square brackets is a *convolution*. [The convolution is due to the phase-difference coupling in (5.1).] As in Section 2, we consider the evolution of a small perturbation $\eta(\theta, t, \omega)$ about the incoherent solution. A miracle occurs when we Fourier analyze the system and calculate the resulting amplitude equations for the different modes of η: the amplitude equations turn out to be uncoupled!

This decoupling occurs because of the convolution form of (5.3) and because the system is being linearized around the incoherent state $\rho_0(\theta, t, \omega) = 1/2\pi$. A simple case of this decoupling phenomenon occurs in the present paper; the evolution of the fundamental mode (Section 3) is separate from that of the higher harmonics (Section 4) precisely because the coupling function $f(\phi) = \sin\phi$ contains no higher harmonics.

For more general $f(\phi)$, the amplitude equations for each mode are very similar to (3.1), and can be analyzed by the methods of Section 3. We find that there is a different critical coupling for each mode. One of the simplest cases occurs if $g(\omega)$ is even and nonincreasing, and $f(\phi)$ is odd, say

$$f(\phi) = \sum_{m=1}^{\infty} b_m \sin(m\phi)$$

Then the critical coupling for mode m is given by

$$K_c(m) = \frac{2}{\pi g(0)}\left(\frac{1}{b_m}\right)$$

which generalizes the classical result (1.4). If $b_m > 0$, then mode m is unstable for $K > K_c(m)$ and neutrally stable for $K < K_c(m)$.

A detailed analysis of (5.1)–(5.3), as well as more general problems including noise terms, will be presented in ref. 19.

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