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The spectrum of the locked state for the Kuramoto model of coupled oscillators

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Abstract

We analyze the linear stability of the phase-locked state in the Kuramoto model of coupled oscillators. The main result is the first rigorous characterization of the spectrum and its associated eigenvectors, for any finite number of oscillators. All but two of the eigenvalues are negative, and merge into a continuous spectrum as the number of oscillators tends to infinity. One eigenvalue is always zero, by rotational invariance. The final eigenvalue, corresponding to a collective mode, determines the stability of the locked state.

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1. Introduction

Thirty years ago, Yoshiki Kuramoto introduced a mathematical model of coupled oscillators that has now become a classic [1–3]. The question that motivated his work comes from biology: how can a large population of interacting oscillators synchronize itself, despite the inevitable diversity in the natural frequencies of the individuals? The phenomenon is ubiquitous in the living world. Examples range from chorusing crickets and fireflies that flash in unison, to the synchronous firing of pacemaker cells in the heart and neurons in the brain [4–7].

But the underlying mathematical basis for collective synchronization had long remained mysterious. Wiener [4] had first posed the problem in the late 1950s, and emphasized its importance for all of science, but lacked the tools

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to solve it. The difficulty was that the oscillators he was considering are nothing like the simple harmonic oscillators used in physics and engineering; rather, they are self-sustained oscillators, with stable limit cycles, and therefore must be nonlinear. The theory of such limit-cycle oscillators was well developed if only one or two were involved, but making sense of the dynamics of hundreds or thousands seemed totally out of reach, especially if they are also non-identical. A few years later, Winfree [5] considered the same puzzle and found resourceful ways to approach it. Using a combination of benchtop experiments with neon tube oscillators, digital simulations, and heuristic mathematics, Winfree discovered that as the variance in the population is gradually reduced, synchronization occurs suddenly and cooperatively, in a manner strikingly reminiscent of a second-order phase transition.

Kuramoto took the crucial next step. In a pioneering 1975 paper [1], just slightly over two pages long, he refined Winfree's model to make it mathematically tractable. Indeed, Kuramoto's analysis of his model (now often called "the Kuramoto model") was a tour de force of insight and ingenuity. It elucidated the phase transition at the onset of synchronization, where the first few oscillators begin to lock their frequencies together. But more generally, and more importantly, it demonstrated that large systems of limit-cycle oscillators could be solvable—in fact, beautifully so. By marrying the mean-field method of equilibrium statistical mechanics to the techniques of nonlinear dynamics, one could make progress on problems that had previously seemed hopeless. In this sense, Kuramoto's work on synchronization was a breakthrough for all of nonlinear science.

In his later writings on this problem [2,3], Kuramoto expressed himself at greater length, giving the reader the opportunity to watch his mind at work. This was a gift to the scientific community. His intellectual openness made a deep impression on both of us at a formative stage in our careers. In 1986, when we were both fresh out of graduate school, we spent many hours reading through Kuramoto's book [2], which was then brand new, admiring not only his brilliance, but also his generosity and his humility. By drawing attention to the unresolved issues in his own arguments, Kuramoto guided us (and many other readers) toward wonderful questions that would have been easy to miss otherwise.

For example, take his discussion of the stability of the incoherent state, in which all the oscillators are desynchronized and uniformly distributed in phase. Having shown that this state bifurcates supercritically at the onset of synchronization, he notes that one would naturally expect it to be stable below the bifurcation, and unstable above it. But then he writes, "Surprisingly enough, this seemingly obvious fact seems difficult to prove" and then explains why [2, p.74]. Indeed, the stability analysis for all aspects of the Kuramoto model has been a rich source of problems for many researchers over the intervening years (for reviews, see [8,9]).

In this paper, we revisit the theme of stability, but now for the end stage of synchronization, not its onset. The question is to analyze the stability of the fully locked state for the Kuramoto model, for any finite number of oscillators. We also give the first detailed characterization of the eigenvalues and eigenvectors for the linearization about the locked state.

Results along these lines have been obtained previously. Ermentrout [10] was the first to find the critical coupling at which all the oscillators become phase-locked. His argument holds for the limit of an infinite number of oscillators, and is based on an existence condition for the locked state, not a stability condition (though stability was checked by numerical simulations). van Hemmen and Wreszinski [11] constructed a Lyapunov function to prove that the phase-locked state is stable for sufficiently large coupling, but they sidestepped the calculation of the spectrum in the general case, since, as they put it, "Figuring out what the spectrum of [the Jacobian] exactly looks like is quite hairy" [11, p.158]. Jadbabaie et al. [12] recently studied phase-locking on graphs of arbitrary connectivity, and computed rigorous bounds for the critical coupling at which locking occurs, using tools from graph theory and control theory.

Before beginning the analysis, we would like to offer a few more words of appreciation about Professor Kuramoto. Although we have only met him on one occasion (the 1997 SIAM Meeting on Dynamical Systems held at Snowbird, Utah), we feel like we have come to know him, at least a little, through his books and papers, his e-mail and, most touchingly, his handwritten correspondence. On so many occasions we have been moved by his kindness—a thoughtful note of condolence after learning of the death of one of our parents, a concerned e-mail after September 11—and by his collegiality in sharing his scientific ideas. Time after time, he has shown us the way, helping us when we were stuck, always gently steering us back on track. Thank you, Professor Kuramoto, our teacher.

2. Model

The Kuramoto model is

$$\dot{\theta}_i = \omega_i + \frac{K}{n}\sum_{j=1}^{n} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, n, \tag{1}$$

where $\theta_i(t)$ is the phase of the ith oscillator at time t, ω_i its natural frequency, and $K > 0$ the coupling strength. If the frequencies ω_i have mean Ω, we can go into a moving frame at frequency Ω to transform (1) to a system where the frequencies have mean 0. So we can assume $\Omega = 0$ without loss of generality; then fixed points of (1) correspond to phase-locked solutions in the original reference frame. We also assume that some $\omega_i \neq 0$ (and so $n \geq 2$). The right hand side of (1) defines the components X_i of a vector field X on the n-fold torus T^n, which is the natural state space for the system (1). Following [11] we have $X = -\nabla F$ where F is the function

$$F(\theta_1, \ldots, \theta_n) = -\sum_{i=1}^{n} \omega_i\theta_i - \frac{K}{2n}\sum_{i,j=1}^{n} \cos(\theta_j - \theta_i).$$

This function is multivalued on the torus, so strictly speaking $X = -\nabla F$ on $\mathbb{R}^n$, but X is only locally a gradient vector field on T^n. Also note that F and X_i are invariant under the 1-parameter group of transformations $\theta_i \mapsto \theta_i + c$.

Introduce the order parameter

$$r\,\mathrm{e}^{\mathrm{i}\psi} = \frac{1}{n}\sum_{j=1}^{n} \mathrm{e}^{\mathrm{i}\theta_j}. \tag{2}$$

If we think of a state $(\theta_1, \ldots, \theta_n)$ as an ordered set of n points $\mathrm{e}^{\mathrm{i}\theta_j}$ on the unit circle, then $r\,\mathrm{e}^{\mathrm{i}\psi}$ is just the centroid of this configuration. Using (2) and a little trig we can rewrite the governing equations as

$$\dot{\theta}_i = X_i = \omega_i + Kr\sin(\psi - \theta_i), \quad i = 1, \ldots, n$$

and

$$F(\theta_1, \ldots, \theta_n) = -\sum_{i=1}^{n} \omega_i\theta_i - \frac{1}{2}Knr^2.$$

3. Locked states

We want to find and classify the locked states, i.e. fixed points for this system. So we try to solve the equations $X_i = 0$. Since some $\omega_i \neq 0$, locked states must have $r > 0$. By invariance, we can set $\psi = 0$ without loss of generality. Then $X_i = 0$ is equivalent to

$$\sin\theta_i = \frac{\omega_i}{Kr},$$

which implies

$$\cos\theta_i = \pm\sqrt{1-\frac{\omega_i^2}{(Kr)^2}}.$$

This gives a self-consistency equation

$$r = \frac{1}{n}\sum_{i=1}^{n} \pm\sqrt{1-\frac{\omega_i^2}{(Kr)^2}} \tag{3}$$

or

$$r^2 = \frac{1}{n}\sum_{i=1}^{n} \pm\sqrt{r^2-\frac{\omega_i^2}{K^2}}.$$

Conversely, any solution to Eq. (3) with $r > 0$ and a given choice of $\pm$ signs defines a locked state $(\theta_1, \dots, \theta_n)$ with order parameter r, since the mean of the ω_i is assumed to be 0. There are 2^n possible choices of signs in this equation, although not all choices yield solutions since the right hand side could be negative. The important point here is that any candidate solution for a stable locked state must, at the very least, have all $\pm$ signs positive. To see this, look at

$$\begin{aligned}\frac{\partial^2 F}{\partial\theta_i^2} &= \frac{K}{n}\sum_{j\neq i}\cos(\theta_j-\theta_i) = \frac{K}{n}\left(\left(\sum_{j=1}^{n}\cos(\theta_j-\theta_i)\right)-1\right) = \frac{K}{n}(rn\cos(\psi-\theta_i)-1)\\ &= Kr\cos(\psi-\theta_i)-\frac{K}{n}.\end{aligned}$$

A necessary condition for stability is that the fixed point must satisfy $\frac{\partial^2 F}{\partial\theta_i^2} \geq 0$ for all i; if $\psi = 0$ this gives

$$r\cos\theta_i \geq \frac{1}{n}$$

and so we must have $\cos\theta_i \geq 0$ for all i. Henceforth we restrict our attention to these locked states. As we will see, some of these are indeed stable and some are not.

The self-consistency equation is equivalent to

$$Kr^2 = \frac{1}{n}\sum_{i=1}^{n}\sqrt{(Kr)^2-\omega_i^2}.$$

Let $x = (Kr)^2$ and

$$f(x) = \frac{1}{n}\sum_{i=1}^{n}\sqrt{x-\omega_i^2}.$$

Fig. 1 plots the graph of the function f. Its domain is $[\omega^2, \infty)$, where $\omega = \max|\omega_i|$. We have $f'(x) > 0$, $f''(x) < 0$ on (ω^2, ∞) and $f'(\omega^2+) = +\infty$. Also, note that $f(\omega^2) > 0$ unless all $|\omega_i|$ are equal. This can only happen if

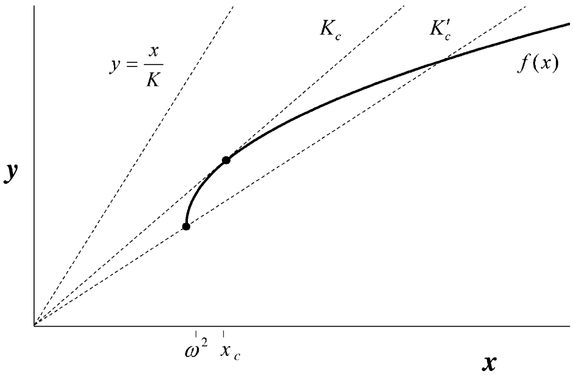

Fig. 1. Schematic plot showing the solution of the self-consistency equation. The line $y = x/K$ is tangent to the graph of $y = f(x)$ when $K = K_c$. At $K = K_c'$, the line passes through the graph at its endpoint minimum. (*Note*: the curvature of the graph is exaggerated here for clarity; in reality, K_c and K_c' can be almost indistinguishable. For example, when $n = 100$ oscillators and the frequencies ω_i are evenly spaced on $[-1, 1]$ such that $\omega_i = -1 + 2(i-1)/(n-1)$, $i = 1, \ldots, n$, we find $K_c \approx 1.285$ and $K_c' \approx 1.287$.)

n is even, $n/2$ of the $\omega_i = \omega$ and the other $n/2$ of the $\omega_i = -\omega$. We will call this the "degenerate case"; here $f(\omega^2) = 0$.

Fig. 1 also shows how to solve the self-consistency equation. Rewriting this equation as $f(x) = K^{-1}x$, we see from the shape of the graph $y = f(x)$ that there exist two critical values of K, $K_c < K_c'$, such that:

$$0 < K < K_c \Rightarrow f(x) = K^{-1}x \text{ has no solutions}, \qquad K = K_c \Rightarrow f(x) = K^{-1}x \text{ has 1 solution},$$
$$K_c < K \leq K_c' \Rightarrow f(x) = K^{-1}x \text{ has 2 solutions}, \qquad K_c' < K \Rightarrow f(x) = K^{-1}x \text{ has 1 solution}$$

(for the degenerate case mentioned above, we have $K_c' = +\infty$).

Now we can parametrize all solutions as follows: think of the ω_i as fixed and let $x \in [\omega^2, \infty)$. (For the degenerate case, consider only $x > \omega^2$.) Then x determines a solution (r, K) to the self-consistency equation with

$$r = \frac{f(x)}{\sqrt{x}}, \qquad K = \frac{x}{f(x)}.$$

Now

$$r = \frac{1}{n}\sum_{i=1}^{n}\sqrt{1 - \frac{\omega_i^2}{x}}$$

and so $\frac{\mathrm{d}r}{\mathrm{d}x} > 0$. We also have

$$\frac{\mathrm{d}K}{\mathrm{d}x} = \frac{f(x) - xf'(x)}{f(x)^2}$$

which is positive if and only if

$$f'(x) < \frac{f(x)}{x}.$$

There is a unique critical value x_c such that

$$f'(x_c) = \frac{f(x_c)}{x_c};$$

i.e. the line joining $(x_c, f(x_c))$ to $(0, 0)$ is tangent to $y = f(x)$, as shown in Fig. 1. For $x > x_c$, $f'(x) < \frac{f(x)}{x}$ and so $\frac{\mathrm{d}K}{\mathrm{d}x} > 0$. Thus (x_c, ∞) parametrizes a branch of solutions to the self-consistency equation with

$$r > r_c = \frac{f(x_c)}{\sqrt{x_c}}, \qquad K > K_c = \frac{x_c}{f(x_c)}.$$

In the next section, we will show that the stable locked states are exactly the states along this branch (whereas the states with $\omega^2 < x < x_c$ are unstable). Note for future reference that as long as $x > \omega^2$,

$$\cos\theta_i = \sqrt{1 - \frac{\omega_i^2}{(Kr)^2}} = x^{-1}\sqrt{x - \omega_i^2} > 0.$$

4. Stability

We now analyze the stability of the locked states, and prove that the locked states with $r > r_c$ described above are attracting. The linearization of (1) at a locked state $(\theta_1, \ldots, \theta_n)$ is given by the equations

$$\dot{x}_i = \frac{K}{n}\sum_{j=1}^{n}\cos(\theta_j - \theta_i)(x_j - x_i), \quad i = 1, \ldots, n.$$

This system is governed by the matrix $J = [J_{ij}]$ where

$$J_{ij} = \frac{\partial X_i}{\partial \theta_j} = -\frac{\partial^2 F}{\partial\theta_i \partial\theta_j} = \frac{K}{n}\cos(\theta_j - \theta_i) \quad \text{for } i \neq j$$

$$= -\frac{\partial^2 F}{\partial\theta_i^2} = \frac{K}{n} - Kr\cos\theta_i \quad \text{for } i = j.$$

The matrix J is symmetric, and so there exists an orthogonal basis of eigenvectors for J.

To characterize the eigenvalues, first observe that

$$\sum_{j=1}^{n} J_{ij} = \left(\frac{K}{n}\sum_{j=1}^{n}\cos(\theta_j - \theta_i)\right) - Kr\cos\theta_i = Kr\cos\theta_i - Kr\cos\theta_i = 0.$$

Thus $(1, \ldots, 1)$ is an eigenvector for J with eigenvalue $\lambda = 0$; this corresponds to the invariance of (1) under the transformations $\theta_i \mapsto \theta_i + c$. The stability condition is that the remaining eigenvalues of J (which are real since J is symmetric) are all negative.

Express

$$J = \frac{K}{n}C - KrD,$$

where

$$C_{ij} = \cos(\theta_j - \theta_i)$$

and D is the diagonal matrix with entries $\cos\theta_i$ along the diagonal. For convenience we put $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$; then

$$C_{ij} = c_i c_j + s_i s_j.$$

Since J is symmetric, the stability condition is equivalent to

$$J\mathbf{x} \cdot \mathbf{x} < 0$$

for all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $\|\mathbf{x}\| > 0$ and $\sum_{i=1}^n x_i = 0$ (i.e. $\mathbf{x}$ is orthogonal to $(1, \ldots, 1)$). With this notation we have

$$J\mathbf{x} \cdot \mathbf{x} = \frac{K}{n}\left[\left(\sum_{i=1}^n c_i x_i\right)^2 + \left(\sum_{i=1}^n s_i x_i\right)^2\right] - Kr\sum_{i=1}^n c_i x_i^2,$$

which can be rewritten more informatively as

$$J\mathbf{x} \cdot \mathbf{x} = \frac{K}{n}\left(\sum_{i=1}^n s_i x_i\right)^2 - Kr\sum_{i=1}^n c_i\left(x_i - \frac{\sum_{j=1}^n c_j x_j}{rn}\right)^2.$$

To check this identity, set $\mu = \sum_{i=1}^n c_i x_i$ and use $\sum_{i=1}^n c_i = rn$ to expand

$$\sum_{i=1}^n c_i\left(x_i - \frac{\mu}{rn}\right)^2 = -\frac{\mu^2}{rn} + \sum_{i=1}^n c_i x_i^2.$$

By rewriting $J\mathbf{x} \cdot \mathbf{x}$ in this way, it becomes apparent that the matrix J can have at most one positive eigenvalue. For if there were two or more, the span of their associated eigenvectors would define at least a two-dimensional subspace on which $J\mathbf{x} \cdot \mathbf{x}$ is positive definite. In that subspace, one could then find a non-zero vector $\mathbf{x}$ orthogonal to the sine vector $(s_1, \ldots, s_n)$. But this is a contradiction: the identity above shows that $J\mathbf{x} \cdot \mathbf{x} \leq 0$ for all vectors orthogonal to the sine vector.

To extract more refined information about the eigenvalues, we now introduce a change of coordinates that diagonalizes the system. Assume for the moment that all $c_i = \cos\theta_i > 0$ (this holds for all solutions except in the endpoint case when $x = \omega^2$). Now change variables to $\mathbf{y} = (y_1, \ldots, y_n)$ where $y_i = \sqrt{c_i} x_i$; then

$$J\mathbf{x} \cdot \mathbf{x} = \frac{K}{n} \left[\left(\sum_{i=1}^{n} \sqrt{c_i} y_i \right)^2 + \left(\sum_{i=1}^{n} \frac{s_i}{\sqrt{c_i}} y_i \right)^2 \right] - Kr\|\mathbf{y}\|^2.$$

The right hand side is the quadratic form

$$Q(\mathbf{y}) = \frac{K}{n}((\mathbf{u} \cdot \mathbf{y})^2 + (\mathbf{v} \cdot \mathbf{y})^2) - Kr\|\mathbf{y}\|^2,$$

where $\mathbf{u} = \left(\sqrt{c_1}, \ldots, \sqrt{c_n}\right)$ and $\mathbf{v} = \left(\frac{s_1}{\sqrt{c_1}}, \ldots, \frac{s_n}{\sqrt{c_n}}\right)$. Observe that $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} s_i = 0$; this is the advantage of the new coordinate system. We can express any $\mathbf{y}$ in the form

$$\mathbf{y} = a\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) + b\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) + \mathbf{y}^{\perp},$$

where $\mathbf{y}^{\perp} \cdot \mathbf{u} = \mathbf{y}^{\perp} \cdot \mathbf{v} = 0$. Then

$$Q(\mathbf{y}) = \frac{K}{n}(\|\mathbf{u}\|^2 a^2 + \|\mathbf{v}\|^2 b^2) - Kr(a^2 + b^2 + \|\mathbf{y}^{\perp}\|^2).$$

Now

$$\|\mathbf{u}\|^2 = \sum_{i=1}^{n} c_i = rn,$$

so we have diagonalized Q:

$$Q(\mathbf{y}) = \frac{K}{n}(\|\mathbf{v}\|^2 - nr)b^2 - Kr\|\mathbf{y}^{\perp}\|^2.$$

Hence the stability condition is

$$\|\mathbf{v}\|^2 - nr < 0$$

or

$$\frac{1}{n} \sum_{i=1}^{n} \frac{s_i^2}{c_i} - r < 0.$$

This is equivalent to

$$\frac{1}{n} \sum_{i=1}^{n} \frac{\omega_i^2}{\sqrt{x - \omega_i^2}} - K^{-1} x < 0;$$

but since $f(x) = K^{-1}x$ we have

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\omega_i^2}{\sqrt{x-\omega_i^2}} - K^{-1}x = \frac{1}{n}\sum_{i=1}^{n}\frac{\omega_i^2 - x}{\sqrt{x-\omega_i^2}} + \frac{1}{n}\sum_{i=1}^{n}\frac{x}{\sqrt{x-\omega_i^2}} - f(x) = -f(x) + 2xf'(x) - f(x)$$

$$= 2x\left(f'(x) - \frac{f(x)}{x}\right) = 2x(f'(x) - K^{-1}).$$

So the stability condition is just $f'(x) < K^{-1}$, which holds for the locked states with $x > x_c$; hence these states are stable. If $\omega^2 < x < x_c$ we have $f'(x) > K^{-1}$, which means that Q (and hence J) has exactly one positive eigenvalue; the remaining eigenvalues are all negative except for one zero eigenvalue corresponding to the invariance of (1). And in the critical case $x = x_c$ we just have $Q(\mathbf{y}) = -Kr\|\mathbf{y}^{\perp}\|^2$, so J has 0 as a double eigenvalue and all remaining eigenvalues are negative. This completes the stability analysis when $x > \omega^2$.

In the limiting case where a solution to the self-consistency equation occurs at the endpoint $x = \omega^2$, some (but not all) of the cosine terms $c_i = \cos\theta_i$ are 0. Suppose that $c_1, \ldots, c_m = 0$ but $c_{m+1}, \ldots, c_n > 0$. As we saw earlier, J can have at most one positive eigenvalue; on the other hand if we put $x_1 = 1$ and all other $x_i = 0$, then

$$J\mathbf{x}\cdot\mathbf{x} = \frac{K}{n} > 0$$

and so there must be at least one positive eigenvalue. Now let us find the dimension of the kernel of J. The equations for the kernel are

$$\left(\sum_{j=1}^{n} c_j x_j\right) c_i + \left(\sum_{j=1}^{n} s_j x_j\right) s_i = nrc_i x_i.$$

Since $c_1 = 0$, $s_1 = \pm 1$ and so we must have $\sum_{i=1}^{n} s_i x_i = 0$ on $\ker J$. Consider the subspace $V \subset \ker J$ given by $\sum_{i=1}^{n} c_i x_i = 0$. Then V has codimension 1 since the vector $(1, \ldots, 1) \in \ker J$ but $\sum_{i=1}^{n} c_i = nr > 0$. Now $(x_1, \ldots, x_n) \in V$ exactly when $x_{m+1} = \cdots = x_n = 0$ and $\sum_{i=1}^{n} s_i x_i = 0$; hence V has dimension $m-1$, and so $\ker J$ has dimension m. So we see that if m of the cosines c_i are 0, then J has one positive eigenvalue, 0 as an eigenvalue with multiplicity m and $n-m-1$ negative eigenvalues (counted of course with multiplicity).

5. Location of eigenvalues

We can exploit the symmetry of the matrix J to get further information on the location of its eigenvalues. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of J. Order the ω_i so that $|\omega_1| \geq \cdots \geq |\omega_n|$; then $-KrD$ has eigenvalues $-Krc_1 \geq \cdots \geq -Krc_n$. We have

$$J\mathbf{x}\cdot\mathbf{x} = \frac{K}{n}C\mathbf{x}\cdot\mathbf{x} - KrD\mathbf{x}\cdot\mathbf{x}$$

and since

$$C\mathbf{x}\cdot\mathbf{x} = \left(\sum_{i=1}^{n} c_i x_i\right)^2 + \left(\sum_{i=1}^{n} s_i x_i\right)^2 \geq 0$$

we must have

$$\lambda_1 \geq -Krc_1, \ldots, \lambda_n \geq -Krc_n$$

by the Courant minimax principle.

We can also derive inequalities in the opposite direction as follows. Let $V \subset \mathbb{R}^n$ be the codimension-2 subspace defined by

$$\sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{n} s_i x_i = 0$$

and let $\mu_1 \geq \cdots \geq \mu_{n-2}$ be the eigenvalues of the quadratic form $J\mathbf{x} \cdot \mathbf{x}$ restricted to V. Since $J\mathbf{x} = -KrD\mathbf{x}$ on V, another application of the minimax principle gives

$$-Krc_1 \geq \mu_1 \geq \lambda_3, \ldots, -Krc_{n-2} \geq \mu_{n-2} \geq \lambda_n.$$

Hence

$$-Krc_i \leq \lambda_i \leq -Krc_{i-2}$$

for $i = 3, \ldots, n-2$, and $-Krc_2 \leq \lambda_2$, $-Krc_1 \leq \lambda_1$. In the case where all $c_i > 0$ this gives an alternate proof that all but two of the eigenvalues λ_i are negative. One of the remaining eigenvalues λ_1, λ_2 is always 0, and the sign of the other determines the stability of the associated locked state, as we saw in the previous section.

6. Characteristic equation

The most detailed information on the eigenvalues comes from the characteristic equation for J. To compute the characteristic polynomial, we express

$$\lambda I - J = \lambda I + KrD - \frac{K}{n}C = (\lambda I + KrD)\left(I - \frac{K}{n}(\lambda I + KrD)^{-1}C\right)$$

and so the characteristic polynomial for J is

$$p(\lambda) = \det(\lambda I + KrD)\det\left(I - \frac{K}{n}(\lambda I + KrD)^{-1}C\right).$$

We will call the first factor $q(\lambda)$; so

$$q(\lambda) = \det(\lambda I + KrD) = \prod_{i=1}^{n}(\lambda + Krc_i).$$

The second factor is not too hard to compute, since the rank of C is only 2. Here is one way to do this that exploits the relations $C_{ij} = c_i c_j + s_i s_j$. Consider the vectors $\mathbf{c} = (c_1, \ldots, c_n)$ and $\mathbf{s} = (s_1, \ldots, s_n)$. Our assumptions imply that

c and **s** are non-zero. Furthermore $\sum_{i=1}^{n} c_i = rn > 0$ and $\sum_{i=1}^{n} s_i = 0$, so **c** and **s** are in fact linearly independent. The matrix $(\lambda I + KrD)^{-1}C$ is conjugate to $C(\lambda I + KrD)^{-1}$, and we have

$$\frac{K}{n}C(\lambda I + KrD)^{-1}\mathbf{c} = R_c(\lambda)\mathbf{c} + R_a(\lambda)\mathbf{s}, \qquad \frac{K}{n}C(\lambda I + KrD)^{-1}\mathbf{s} = R_a(\lambda)\mathbf{c} + R_s(\lambda)\mathbf{s},$$

where the rational functions $R_i(\lambda)$ are defined to be

$$R_c(\lambda) = \frac{K}{n}\sum_{i=1}^{n}\frac{c_i^2}{\lambda + Krc_i}, \tag{4}$$

$$R_s(\lambda) = \frac{K}{n}\sum_{i=1}^{n}\frac{s_i^2}{\lambda + Krc_i}, \tag{5}$$

$$R_a(\lambda) = \frac{K}{n}\sum_{i=1}^{n}\frac{c_i s_i}{\lambda + Krc_i}. \tag{6}$$

The determinant of this 2×2 system is

$$\begin{aligned} R_c(\lambda)R_s(\lambda) - R_a(\lambda)^2 &= \frac{K^2}{n^2}\sum_{i,j=1}^{n}\frac{c_i^2 s_j^2 - c_i s_i c_j s_j}{(\lambda + Krc_i)(\lambda + Krc_j)} = \frac{K^2}{2n^2}\sum_{i,j=1}^{n}\frac{c_i^2 s_j^2 + c_j^2 s_i^2 - 2c_i s_i c_j s_j}{(\lambda + Krc_i)(\lambda + Krc_j)} \\ &= \frac{K^2}{n^2}\sum_{i<j}\frac{(c_i s_j - c_j s_i)^2}{(\lambda + Krc_i)(\lambda + Krc_j)} \end{aligned}$$

which is generically non-zero, since $(c_i s_j - c_j s_i)^2 = \sin^2(\theta_j - \theta_i)$, and these coefficients cannot all be 0. Therefore $C(\lambda I + KrD)^{-1}$ maps the span of **c** and **s** onto itself, and the span of **c** and **s** intersects the kernel of $C(\lambda I + KrD)^{-1}$ only at **0**. Now $C(\lambda I + KrD)^{-1}$ has rank 2, so we can extend **c** and **s** to a basis of $\mathbb{R}^n$ where the remaining $n-2$ basis vectors span the kernel of $C(\lambda I + KrD)^{-1}$. The matrix of $\frac{K}{n}C(\lambda I + KrD)^{-1}$ with respect to this basis is

$$\begin{pmatrix} R_c(\lambda) & R_a(\lambda) & 0 & \cdots & 0 \\ R_a(\lambda) & R_s(\lambda) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and so we have

$$\det\left(I - \frac{K}{n}(\lambda I + KrD)^{-1}C\right) = (1 - R_c(\lambda))(1 - R_s(\lambda)) - R_a(\lambda)^2$$

which gives

$$p(\lambda) = q(\lambda)[(1 - R_c(\lambda))(1 - R_s(\lambda)) - R_a(\lambda)^2].$$

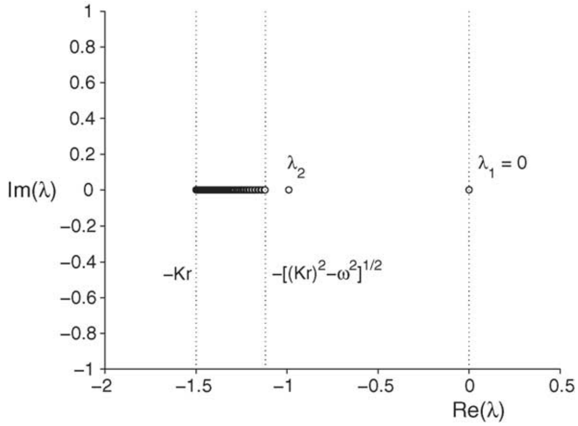

Fig. 2. Spectrum of the stable locked state. All but two of the eigenvalues lie between $-Kr$ and $-\sqrt{(Kr)^2-\omega^2}$. As n tends to infinity, these eigenvalues merge into a continuous spectrum. The largest eigenvalue $\lambda_1 = 0$, by rotational invariance. The second largest eigenvalue, λ_2, is strictly negative, and corresponds to the non-trivial part of the discrete spectrum as $n \to \infty$. This eigenvalue determines the stability of the locked state. In the example shown here, K is chosen so that $Kr = 1.5$; $n = 100$ oscillators; and the frequencies are evenly spaced: $\omega_i = -1 + 2(i-1)/(n-1)$, $i = 1, \ldots, n$, with $\omega = \max|\omega_i| = 1$.

The function $R_a(\lambda)$ is related to the asymmetry of the configuration of angles θ_i (hence the subscript). In fact $R_a(\lambda)$ is identically 0 if and only if the non-zero frequencies ω_i can be grouped in plus–minus pairs $\omega_{2i} = -\omega_{2i-1}$, so the sine terms s_{2i-1} and s_{2i} cancel (the equivalence holds even if some of the $c_i = 0$, because we also have $\sum_{i=1}^n s_i = 0$).

This symmetric case is very easy to analyze. We have

$$p(\lambda) = q(\lambda)(1 - R_c(\lambda))(1 - R_s(\lambda)).$$

The key point is that away from their poles, the functions $R_c(\lambda)$ and $R_s(\lambda)$ are strictly decreasing, and so the equations $R_c(\lambda) = 1$ and $R_s(\lambda) = 1$ must each have exactly one solution between any two consecutive poles.

Fig. 2 shows a typical spectrum for this symmetric case, and Figs. 3 and 4 plot the associated rational functions $R_c(\lambda)$ and $R_s(\lambda)$. The spectrum is made up of three pieces: a heavily populated part destined to become continuous spectrum as $n \to \infty$; a trivial eigenvalue at 0, corresponding to the rotational symmetry of the Kuramoto model; and a lone eigenvalue in between those two, destined to become the non-trivial part of the discrete spectrum as $n \to \infty$.

To understand why the spectrum has this appearance, realize first that all $c_i > 0$ for the stable locked state being considered here. Furthermore, $R_c(0) = 1$ and $q(0) \neq 0$, so $q(\lambda)(1 - R_c(\lambda))$ has a simple zero at $\lambda = 0$. The equation $R_s(\lambda) = 1$ has exactly one root λ such that $\lambda > -Krc_i$ for all i; since $R_s(\lambda)$ is decreasing, this root is negative if and only if $R_s(0) < 1$, which gives the stability condition

$$\frac{1}{n}\sum_{i=1}^{n}\frac{s_i^2}{c_i} < r$$

that we derived earlier. Thus the two largest eigenvalues λ_1, λ_2 satisfy $\lambda > -Krc_i$ for all i; one of them is 0, and the other is negative if and only if the stability condition above holds. If we have m of the $c_i = 0$, then $R_s(\lambda)$ has a simple pole and $1 - R_c(\lambda)$ has a simple zero at $\lambda = 0$, so we have one positive eigenvalue, and the eigenvalue 0 has multiplicity m, again just as we saw earlier.

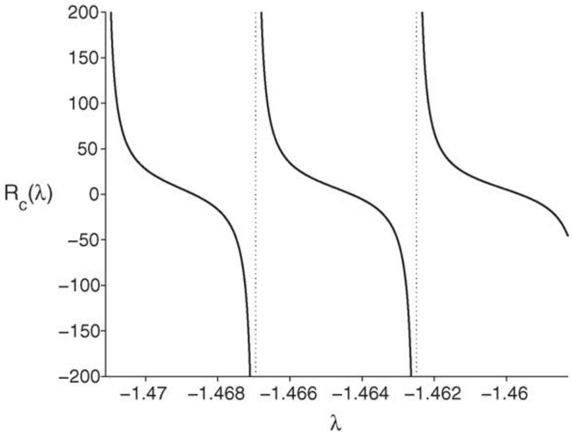

Fig. 3. Graph of $R_c(\lambda)$ from Eq. (4), for the neighborhood of the eigenvalue shown in Fig. 5. Eigenvalues associated with even eigenvectors satisfy $R_c(\lambda) = 1$. A unique root exists between every two poles of this rational function.

The remaining negative eigenvalues are also easy to locate. If all $\omega_i \neq 0$, then n is even and the ω_i come in $+/-$ pairs. The equations $R_c(\lambda) = 1$ and $R_s(\lambda) = 1$ both have exactly one root between any two consecutive poles, so we have two eigenvalues between each pair of consecutive distinct values of $-Krc_i$. If the ω_i are all distinct, this locates $n - 2$ negative eigenvalues, which together with λ_1 and λ_2 accounts for all the eigenvalues. If ω_i occurs with multiplicity m_i, then $-Krc_i$ occurs with multiplicity $2m_i$ as a root of $q(\lambda)$; $R_c(\lambda)$ and $R_s(\lambda)$ have simple poles at $\lambda = -Krc_i$, and so $\lambda = -Krc_i$ is an eigenvalue of J with multiplicity $2(m_i - 1)$. Again together with λ_1 and λ_2, this accounts for all the eigenvalues. If $\omega_n = 0$, the only change is that we get just one eigenvalue between $-Krc_n = -Kr$ and the next largest value $-Krc_i$, because $R_s(\lambda) < 0$ on this interval (since $s_n = 0$, $R_s(\lambda)$ does not have a pole at $\lambda = -Kr$). We also can see that if 0 occurs with multiplicity m_n among the ω_i, then $\lambda = -Kr$ is an eigenvalue of J with multiplicity $m_n - 1$, so once again we've located all the eigenvalues of J. Thus when the natural frequencies are symmetrically distributed about 0, we can obtain even sharper estimates of the eigenvalues than we derived earlier using the minimax principle.

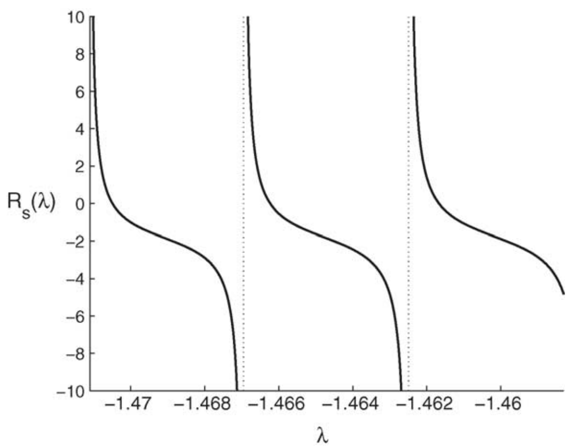

Fig. 4. Graph of $R_s(\lambda)$ from Eq. (5), for λ near the eigenvalue shown in Fig. 6. Eigenvalues associated with odd eigenvectors satisfy $R_s(\lambda) = 1$. A unique root exists between every two poles of this rational function.

The analysis gets a little trickier in the asymmetric case where $R_a(\lambda) \neq 0$. Assume first that all the terms $-Krc_i$ are distinct and non-zero. As in the symmetric case, the equation $(1 - R_c(\lambda))(1 - R_s(\lambda)) = 0$ has two roots in each interval $(-Krc_i, -Krc_{i-1})$, where $2 \leq i \leq n-1$, and also in $(-Krc_1, \infty)$. (When $i = n$ we could have $\omega_n = 0$, so we are only guaranteed one root ν_n.) Label these two roots $\mu_i \leq \nu_i$. The function $(1 - R_c(\lambda))(1 - R_s(\lambda)) - R_a(\lambda)^2$ is negative or zero at the endpoints of the interval $[\nu_i, \mu_{i-1}]$. $q(\lambda)$ changes sign as λ crosses $-Krc_{i-1}$, so the characteristic polynomial $p(\lambda) = q(\lambda)((1 - R_c(\lambda))(1 - R_s(\lambda)) - R_a(\lambda)^2)$ must have a root λ_i somewhere in the interval $[\nu_i, \mu_{i-1}]$, for $i \geq 2$. And since

$$\lim_{\lambda \to \infty} [(1 - R_c(\lambda))(1 - R_s(\lambda)) - R_a(\lambda)^2] = 1,$$

there is one more eigenvalue $\lambda_1 \geq \nu_1$. Hence we obtain the estimates

$$-Krc_i < \lambda_i < -Krc_{i-2}$$

for $i = 3, \ldots, n$ and $-Krc_2 < \lambda_2$, $-Krc_1 < \lambda_1$ which are a bit sharper than our earlier results. This accounts for the $n-2$ negative eigenvalues. We have $R_c(0) = 1$ and $R_a(0) = 0$; hence $p(0) = 0$, so one of λ_1, λ_2 is always 0. And $R_c(0) = 1$ implies one of μ_1, ν_1 is also always 0. The inequality

$$\lambda_2 \leq \mu_1 \leq \nu_1 \leq \lambda_1$$

shows that when $\mu_1 < 0$, $\lambda_2 < 0$ so $\lambda_1 = 0$. Whereas if $\nu_1 > 0$ then $\lambda_1 > 0$ and so $\lambda_2 = 0$. And if $R_s(0) = 0$, so both $\mu_1 = \nu_1 = 0$, then $(1 - R_c(\lambda))(1 - R_s(\lambda)) - R_a(\lambda)^2$ has a double root at 0, so $\lambda_1\lambda_2 = 0$. So the stability is determined by the location of the largest root of the equation $1 - R_s(\lambda) = 0$, and the condition for stability is $R_s(0) < 1$, just as expected.

In the limiting case when $c_1, \ldots, c_m = 0$ and the remaining $c_i > 0$, $\lambda = 0$ is a root of multiplicity m of $q(\lambda)$. $1 - R_c(\lambda)$ has a zero and $1 - R_s(\lambda)$ has a pole at $\lambda = 0$; an easy calculation shows that

$$[(1 - R_c(\lambda))(1 - R_s(\lambda)) - R_s(\lambda)^2]|_{\lambda=0} = -\frac{(n-m)m}{n^2r^2} \neq 0,$$

so $\lambda = 0$ is a root of the characteristic polynomial $p(\lambda)$ of multiplicity m in agreement with our earlier results. But unlike the symmetric case, if a value c_i occurs with multiplicity $m_i > 1$ (which need not be even in this case), we cannot determine the exact multiplicity of the eigenvalue $\lambda = -Krc_i$; however it must be at least $m_i - 1$, since the function $(1 - R_c(\lambda))(1 - R_s(\lambda)) - R_a(\lambda)^2$ has at most a simple pole at $\lambda = -Krc_i$ (the $(\lambda + Krc_i)^{-2}$ terms cancel).

7. Description of eigenvectors

In the symmetric case it is possible to give a simple description of the eigenvectors associated to the eigenvalues λ_i. Order the natural frequencies ω_i so that $|\omega_1| \geq \cdots \geq |\omega_n|$, $\omega_{2i} = -\omega_{2i-1}$ for $i = 1, \ldots, k$ and $\omega_{2k+1} \cdots = \omega_n = 0$. We can split $\mathbb{R}^n$ into two subspaces V_{even} and V_{odd} where $\mathbf{x} = (x_1, \ldots, x_n) \in V_{\text{even}}$ when $x_{2i} = x_{2i-1}$ for $i = 1, \ldots, k$ and $\mathbf{x} \in V_{\text{odd}}$ when $x_{2i} = -x_{2i-1}$ for $i = 1, \ldots, k$ and $x_i = 0$ for $i \geq 2k+1$. Then $\dim V_{\text{even}} = n - k$ and $\dim V_{\text{odd}} = k$. We have $\mathbf{c} \in V_{\text{even}}$ and $\mathbf{s} \in V_{\text{odd}}$; V_{even} and V_{odd} are orthogonal, and most importantly, J preserves the even and odd subspaces. Therefore we can find bases for both V_{even} and V_{odd} consisting of eigenvectors for J. Let us describe these eigenvectors.

If $\mathbf{x} \in V_{\text{even}}$ then

$$J\mathbf{x} = \frac{K}{n}(\mathbf{c} \cdot \mathbf{x})\mathbf{c} - KrD\mathbf{x},$$

so the eigenvalue equation on V_{even} is

$$(\lambda I + KrD)\mathbf{x} = \frac{K}{n}(\mathbf{c} \cdot \mathbf{x})\mathbf{c}.$$

Suppose $\mathbf{c} \cdot \mathbf{x} \neq 0$; then we may as well assume $\mathbf{c} \cdot \mathbf{x} = \frac{n}{K}$, and we obtain

$$x_i = \frac{c_i}{\lambda + Krc_i}$$

(interpret this as $1/Kr$ if λ and c_i are both 0). We must satisfy the self-consistency equation

$$\mathbf{c} \cdot \mathbf{x} = \frac{K}{n}$$

which is equivalent to $R_c(\lambda) = 1$. So we obtain eigenvectors $\mathbf{x}$ with

$$x_i = \frac{\sqrt{(Kr)^2 - \omega_i^2}}{\lambda + \sqrt{(Kr)^2 - \omega_i^2}} \tag{7}$$

for each root of $R_c(\lambda) = 1$.

Fig. 5 plots the components of a typical even eigenvector associated with these eigenvalues. To understand its shape intuitively, think of a locked state as a configuration of points on the unit circle, with centroid on the positive

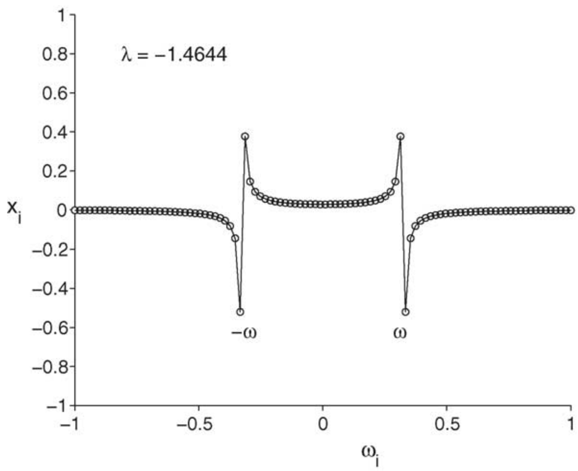

Fig. 5. Components of a typical even eigenvector in the "continuous" part of the spectrum, corresponding to an eigenvalue $\lambda \approx -1.4644$. Parameters as in Fig. 2. In this eigenmode, most of the oscillators remain close to their locked positions. Large displacements occur only for oscillators with natural frequencies near $\omega' = \pm\sqrt{(Kr)^2 - \lambda^2} \approx 0.325$. Open circles, eigenvector components obtained numerically; solid line, Eq. (7).

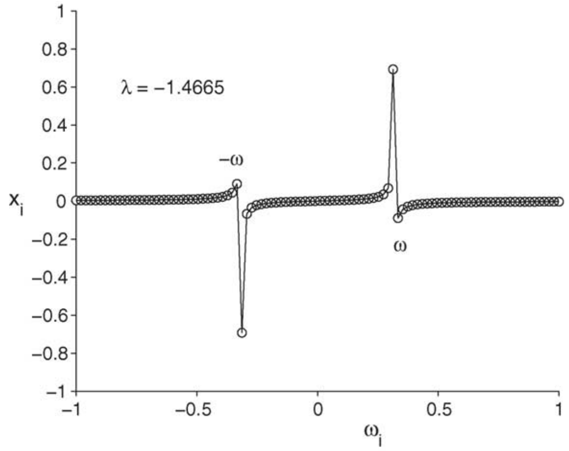

Fig. 6. Components x_i of a typical odd eigenvector in the "continuous" part of the spectrum, corresponding to an eigenvalue $\lambda \approx -1.4665$. Parameters as in Fig. 2. As in Fig. 5, nearly all of the oscillators remain close to their locked positions; large displacements occur only for oscillators with natural frequencies near $\omega' = \pm\sqrt{(Kr)^2 - \lambda^2} \approx 0.315$. Open circles, eigenvector components obtained numerically; solid line, Eq. (8).

real axis. Then these eigenvectors correspond to purely vertical perturbations of the centroid, since the corresponding first-order variation in the centroid has components

$$\nabla(r\cos\psi)\cdot\mathbf{x} = -\frac{1}{n}\sum_{i=1}^{n}(\sin\theta_i)x_i = 0 \quad \text{and} \quad \nabla(r\sin\psi)\cdot\mathbf{x} = \frac{1}{n}\sum_{i=1}^{n}(\cos\theta_i)x_i \neq 0.$$

They include the eigenvector $(1, \ldots, 1)$ with $\lambda = 0$ corresponding to the invariance under rotation in the Kuramoto model; the others all have $\lambda < 0$.

Similarly, if $\mathbf{x} \in V_{\text{odd}}$ the eigenvalue equation is

$$(\lambda I + KrD)\mathbf{x} = \frac{K}{n}(\mathbf{s}\cdot\mathbf{x})\mathbf{s}.$$

Assuming $\mathbf{s}\cdot\mathbf{x} \neq 0$ we get odd eigenvectors $\mathbf{x}$ with

$$x_i = \frac{\omega_i}{\lambda + \sqrt{(Kr)^2 - \omega_i^2}}, \tag{8}$$

where λ satisfies $R_s(\lambda) = 1$. These eigenvectors (a typical example of which is shown in Fig. 6) correspond to horizontal perturbations of the centroid, since here $\nabla(r\cos\psi)\cdot\mathbf{x} \neq 0$ but $\nabla(r\sin\psi)\cdot\mathbf{x} = 0$. They also include the eigenvector shown in Fig. 7, corresponding to the largest root of $R_s(\lambda) = 1$, which determines the stability of the locked state.

The remaining eigenvectors $\mathbf{x}$ satisfy $\mathbf{c}\cdot\mathbf{x} = \mathbf{s}\cdot\mathbf{x} = 0$, and so are also eigenvectors of $-KrD$. These only occur if some of the $|\omega_i|$ occur with multiplicities, and can be described as follows. Suppose $\omega_{2i} = \omega_{2j} \neq 0$, where $j \neq i$. We can construct an even eigenvector $\mathbf{x}$ by setting $x_{2i} = x_{2i-1} = 1$, $x_{2j} = x_{2j-1} = -1$ and all other $x_k = 0$. It is easy to picture this perturbation: we have two oscillators at location θ_{2i} and two at $-\theta_{2i}$; we perturb the two at θ_{2i} by equal amounts in opposite directions, and mirror this in the reflection through the x-axis at $-\theta_{2i}$. To first order the centroid of the configuration remains constant.

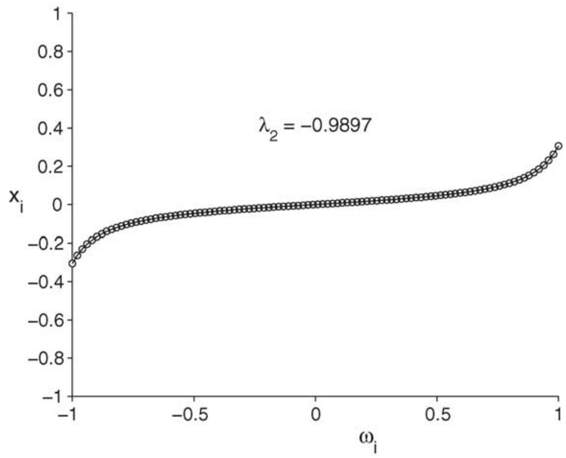

Fig. 7. Components of the odd eigenvector corresponding to the "discrete" part of the spectrum, associated with the second largest eigenvalue $\lambda_2 = -0.9897$. Parameters as in Fig. 2. Open circles, eigenvector components obtained numerically; solid line, Eq. (8). In contrast to the eigenvectors shown in Figs. 5 and 6, this is a collective mode: all the oscillators are significantly displaced from their locked positions.

In the same way, an odd eigenvector $\mathbf{x}$ can be constructed by setting $x_{2i} = x_{2j-1} = 1$, $x_{2i-1} = x_{2j} = -1$. (By the way, these two eigenvectors are identical as perturbations; the even or odd distinction is an artifact of the indexing.) If ω_i occurs with multiplicity m_i, we can construct $2(m_1 - 1)$ linearly independent eigenvectors this way. And if $\omega_n = 0$ with multiplicity $m_n = n - 2k \geq 2$ we get $m_n - 1$ linearly independent even eigenvectors from this construction, but no odd eigenvectors since odd vectors have $x_i = 0$ whenever $\omega_i = 0$. In light of our previous work, we see that this accounts for all the eigenvalues of J. In the degenerate case where all $|\omega_i|$ are equal, we get $n - 2$ eigenvectors of this form; the remaining two correspond to the invariance under rotation $(1, \ldots, 1)$ and the (unique) root of $R_s(\lambda) = 1$.

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