Structure of Long-Term Average Frequencies for Kuramoto Oscillator Systems

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We study the long-term average frequency as a function of the natural frequency for Kuramoto oscillators with periodic coefficients. Unlike the case for more general periodically forced oscillators, this function is never a "devil's staircase"; it may have plateaus at integer multiples of the forcing frequency, but we prove it is strictly increasing between these plateaus. The proof uses the fact that the flow maps for Kuramoto oscillators extend to Möbius transformations on the complex plane, and that Möbius transformations have particularly simple dynamics that rule out p:q mode locking except in the case of fixed points ($q = 1$). We also give a criterion for the degeneration of an integer plateau to a single point and use it to explain the absence of plateaus at even multiples of the collective frequency for a Kuramoto system with a bimodal frequency distribution.

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The Kuramoto coupled oscillator model has played a central role in the study of coupled oscillator systems in physics and biology since it was proposed by Kuramoto in 1975 [1]. The model consists of a population of oscillators with random natural frequencies, all coupled to each other via sinusoidal terms. Kuramoto's model has been studied extensively in both the physics and applied mathematics literature; see [2] for a partial survey through the year 2000. This Letter addresses the dependence of the long-term average frequencies of the oscillators in the model as a function of the oscillators' natural frequencies. As we shall see, the structure of this relationship is completely different than for the case of more general oscillator models with varying natural frequencies.

Generically, i.e., for "most" oscillator models with varying natural frequencies (in a sense which can be made mathematically precise), the dependence of average frequencies as a function of natural frequencies is given by a universal structure known in the dynamical systems literature as a "devil's staircase," shown on page 111 in [3]. The main result of this Letter is that the Kuramoto model can never exhibit this devil's staircase relationship but instead has a much simpler structure. The proof builds on the result, proved in [4], that the evolution in time of any configuration of oscillators with the same natural frequency in the Kuramoto model is given by a Möbius map, i.e., fractional linear transformation of the form $z \mapsto (az + b)/(cz + d)$ on the complex plane, which in this case has the additional property that the map preserves the unit circle.

Moreover, this time evolution under Möbius transformations holds more generally for any periodically forced oscillator with varying natural frequency and only first-order sine and cosine terms. Such models include the overdamped, periodically forced pendulum [5] and a series array of Josephson junctions with purely resistive load [6]. Hence our main result below also extends to these models. Our work demonstrates that general coupled or forced oscillator systems exhibit behavior which does not occur in the highly idealized Kuramoto model, and thus reveals a limitation of the Kuramoto model as a paradigm for general oscillator models.

The classic Kuramoto model is the system

$$\dot{\theta}_j = \omega_j + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j), \tag{1}$$

where $K \geq 0$ is a coupling constant, and ω_j is the natural frequency of the jth oscillator. This system can be simplified by introducing the complex order parameter

$$Z(t) = \frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k(t)}; \tag{2}$$

then (1) reduces to $\dot{\theta}_j = \omega_j + K\mathrm{Im}(Z(t)\exp(-i\theta_j))$, which expresses the evolution of the jth oscillator solely in terms of the order parameter $Z(t)$.

Typically the natural frequencies are chosen according to a probability density $g(\omega)$ with infinite tails, like a Gaussian or Lorentzian density. Then as N gets large the system (1) usually has no fixed points and so cannot be analyzed by doing a linear stability analysis around fixed points. This is why the finite $-N$ Kuramoto model described above is so notoriously difficult. So in the study of the Kuramoto model, one typically considers the infinite $-N$ limit, first proposed in [7], whose states are density functions $\eta(\theta, \omega, t)$ that describe the distribution of the oscillators with frequency ω at time t. The order parameter is now

$$Z(t) = \int_0^{\infty}\int_0^{2\pi} e^{i\theta}\eta(\theta, \omega, t)g(\omega)d\theta d\omega, \tag{3}$$

and the population of oscillators with frequency ω evolves according to the equation

$$\dot{\theta} = \omega + K\mathrm{Im}(Z(t)e^{-i\theta}). \tag{4}$$

Now suppose that the order parameter $Z(t)$ is periodic with period $T > 0$, which does indeed happen for several important cases of the Kuramoto model which we discuss below. Then equation (4) is an example of a periodically forced differential equation of the form

$$\dot{\theta} = \omega + a(t)\cos\theta + b(t)\sin\theta + c(t), \tag{5}$$

where $\omega \in \mathbb{R}$ is a parameter and the functions $a(t)$, $b(t)$, and $c(t)$ are assumed to be continuous with period T. We call this equation a Kuramoto oscillator equation or Kuramoto oscillator. We emphasize that it doesn't matter whether the coefficients $a(t)$, $b(t)$, and $c(t)$ are "external" forcing terms, or averages over some population of oscillators, as in (4); all that matters is the form of the equation and that all oscillators are governed by the same equation with periodic coefficients, except for the variation due to the parameter ω.

In turn, (5) is a special case of the general parameter-dependent periodically forced oscillator equation

$$\dot{\theta} = \omega + \epsilon f(\theta, t), \tag{6}$$

where f is continuous and has period 2π in θ, $T > 0$ in t, ϵ is a coupling constant, and $\omega \in \mathbb{R}$ is a parameter. Henceforth we will consider θ to be a real variable; we do this so we can keep track of the number of times a solution $\theta(t)$ to an equation like (6) laps around the unit circle. We are interested in the long-term average angular frequency of solutions to (6), defined by

$$\rho(\omega) = \lim_{t\to\infty} \frac{\theta(t)}{t}; \tag{7}$$

as we shall see momentarily, this limit exists and depends only on the natural frequency ω.

The time–T flow map $F_\omega: \mathbb{R} \to \mathbb{R}$ for (6) is defined as follows: for any $\theta_0 \in \mathbb{R}$, let $\theta(t)$ be the solution to (6) with initial condition $\theta(0) = \theta_0$, then $F_\omega(\theta_0) = \theta(T)$. From the theory of flows for dynamical systems [8] the map F_ω is continuous, strictly increasing in θ and ω and satisfies $F_\omega(\theta + 2\pi) = F_\omega(\theta) + 2\pi$. A continuous, strictly increasing function $F: \mathbb{R} \to \mathbb{R}$ such that $F(\theta + 2\pi) = F(\theta) + 2\pi$ is called a circle map lift since it induces an orientation-preserving homeomorphism on the unit circle [3]. Any circle map lift F has a rotation number

$$\mathrm{rot}\,(F) = \lim_{n\to\infty} \frac{F^n(\theta_0)}{n} \tag{8}$$

independent of the initial condition θ_0. The rotation number of the map F_ω determines $\rho(\omega)$:

$$\rho(\omega) = \lim_{n\to\infty} \frac{\theta(nT)}{nT} = \lim_{n\to\infty} \frac{F_\omega^n(\theta_0)}{nT} = \frac{1}{T}\,\mathrm{rot}(F_\omega). \tag{9}$$

From the theory of circle maps [3], ρ generically has the familiar devil's staircase structure, consisting of plateaus at all rational multiples p/q (in lowest terms) of $2\pi/T$. The natural frequency ω is in the p/q plateau if there is a stable solution $\theta(t)$ to (6) such that $\theta(t + qT) = \theta(t) + 2\pi p$, so this solution mapped to the circle orbits the circle p times over q cycles of $f(\theta, t)$; this is called stable $p{:}q$ mode locking of solutions to the periodic forcing. The widths of the plateaus shrink to 0 as the coupling $\epsilon \to 0$, in the pattern of Arnold tongues in the (ϵ, ω) plane.

But the structure of the function ρ is vastly simpler for the classic Kuramoto model, or Kuramoto oscillators governed by (5). For example, if the frequency distribution for the ω is Gaussian or Lorentzian with mean 0, and K is above a critical level, then (4) has self-consistent solutions with constant order parameter $Z(t) = R > 0$. The average frequency function ρ has only one plateau, corresponding to phase-locking for small ω (see Fig. 1(a)). This isn't too surprising since in this case there isn't any time dependence in (4), so one would expect the function ρ to be strictly increasing outside the phase-locking plateau.

Now consider a bimodal Gaussian distribution for ω, defined as the average of two Gaussians with identical widths but different means. The order parameter for the Kuramoto system can then undergo a Hopf bifurcation [9], so for appropriately chosen parameters, $Z(t)$ varies periodically. In this case we find the graph of ρ does indeed

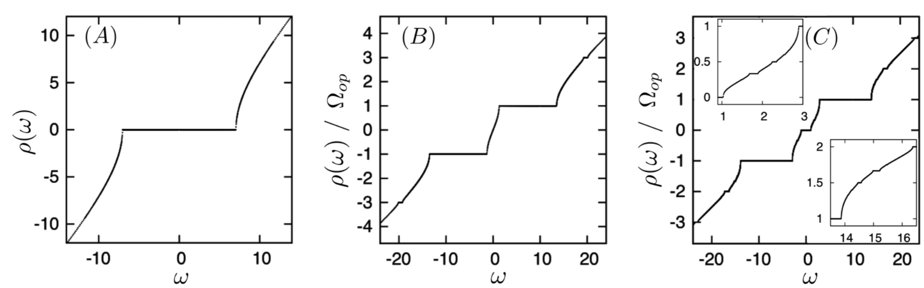

FIG. 1. (A) Average frequency $\rho(\omega)$ vs. natural frequency ω for system (1); $K = 10$, $N = 4096$ and ω_j drawn from a Gaussian with mean 0 and width 5. (B) $\rho(\omega)$ normalized by Ω_{op} for system (1); $K = 21$, $N = 4096$ and ω_j drawn from a bimodal Gaussian with constituent means at $\omega = \pm 10$ with widths 5. Note absence of plateaus at even multiples of Ω_{op}. (C) normalized $\rho(\omega)$ for system (10) with higher-order coupling; $K_1 = 8$, $K_2 = 14$, $N = 4096$ and ω_j drawn from the same bimodal Gaussian distribution as in (B). Note full devil's staircase pattern of plateaus in (C), magnified in insets.

have multiple plateaus, but only at integer multiples of the order parameter frequency $\Omega_{\rm op} = 2\pi/T$, as shown in Fig. 1(b). This is still quite different from the generic devil's staircase. Notice that in this example the plateaus all correspond to odd multiples of $\Omega_{\rm op}$; in particular, there is no plateau centered at $\omega = 0$. We will explain this curious phenomenon in the discussion at the end of this Letter. This pattern of integer-only plateaus was observed by Sakaguchi for the Kuramoto model with an additional periodic forcing term [10,11] and has also been observed in the study of Josephson junctions and certain types of torus flows [12–15]. By contrast, if we add a higher-order coupling term to (1), the system

$$\dot{\theta}_j = \omega_j + \frac{K_1}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j) + \frac{K_2}{N}\sum_{k=1}^{N}\sin 2(\theta_k - \theta_j) \quad (10)$$

has the full devil's staircase structure, shown in Fig. 1(c).

As mentioned earlier, the reason that the function ρ is so different for Kuramoto oscillators as opposed to more general periodically forced oscillators is the time–t flow map for (5) is a Möbius map on the circle, and Möbius maps have very special dynamical behavior. In particular, a Möbius map can never have stable periodic points except for fixed points, which rules out stable p:q mode locking when $q > 1$. The relation of Möbius maps to the Kuramoto system was noted in Ref. [4], and to Josephson junction systems in Ref. [12], where it was used to rule out phase locking with fractional rotation numbers for certain types of Josephson junction models. We extend this result by proving that as a consequence of the dynamical properties of Möbius maps, the function ρ must be strictly increasing outside the integer plateaus; we also prove a precise criterion for the degeneration of an integer plateau to a single point.

Consider the time–T flow map F_ω for the Kuramoto oscillator (5). If $M \geq 0$ is a bound for the periodic term $a(t)\cos\theta + b(t)\sin\theta + c(t)$, then

$$\theta + n(\omega - M)T \leq F_\omega^n(\theta) \leq \theta + n(\omega + M)T \quad (11)$$

for any θ and hence $|\rho(\omega) - \omega| \leq M$. Therefore $\rho(\omega) \sim \omega$ as $\omega \to \pm\infty$. Since ρ is continuous and increasing, for each $k \in \mathbb{Z}$ there is a maximal interval $[\omega_k^-, \omega_k^+]$ on which $\rho(\omega) = 2\pi k/T$; clearly $\omega_k^- \leq \omega_k^+ < \omega_{k+1}^-$ for all $k \in \mathbb{Z}$. We call the intervals $[\omega_k^-, \omega_k^+]$ the integer plateaus for ρ. Our goal is to prove that ρ is strictly increasing on the intervals $[\omega_k^+, \omega_{k+1}^-]$ between the integer plateaus. We shall also prove that the case where the integer plateau $[\omega_k^-, \omega_k^+]$ is just a point is quite exceptional, in a sense to be made precise below.

More generally, consider any family G_λ of circle map lifts that is strictly increasing in λ. The function defined by $\phi(\lambda) = \mathrm{rot}(G_\lambda)$ usually isn't strictly increasing, but there is a special case where we can conclude that ϕ is strictly increasing at some point $\lambda = \lambda_0$:

Lemma. If the family G_λ of circle map lifts is strictly increasing in λ, and G_{λ_0} is conjugate via a circle map lift Φ to a translation on $\mathbb{R}$, then ϕ is strictly increasing at λ_0; in other words, $\{\phi(\lambda) = \phi(\lambda_0)\} = \{\lambda_0\}$.

Proof. The family $\Phi G_\lambda \Phi^{-1}$ of circle map lifts is also strictly increasing in λ and has the same rotation number function as G_λ (Ref. [3]); hence it suffices to consider the case where $G_{\lambda_0}(\theta) = \theta + C$ for some $C \in \mathbb{R}$. We have $C = \phi(\lambda_0)$ since C is the rotation number of the map translation by C. Now fix some $\lambda > \lambda_0$; the periodicity condition on the map G_λ insures the existence of some $\epsilon > 0$ such that

$$G_\lambda(\theta) \geq \theta + \phi(\lambda_0) + \epsilon \quad (12)$$

for all $\theta \in \mathbb{R}$, which in turn implies that

$$G_\lambda^n(\theta) \geq \theta + n(\phi(\lambda_0) + \epsilon) \quad (13)$$

for all n, and hence $\phi(\lambda) \geq \phi(\lambda_0) + \epsilon$, and a similar result holds for $\lambda < \lambda_0$. Therefore ϕ is strictly increasing at $\lambda = \lambda_0$. ■

If we let $z = e^{i\theta}$ in (5) and substitute

$$\dot{\theta} = -i\frac{\dot{z}}{z}, \qquad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right), \quad (14)$$

then (5) transforms into an equation of the form

$$\dot{z} = A(t)z^2 + B(t)z + C(t), \quad (15)$$

which extends to a Riccati equation on the extended complex plane $\hat{\mathbb{C}}$. The exact form of the coefficients isn't important in our analysis; we only need that they have period T and depend continuously on ω. The time–T flow map for a Riccati equation is a Möbius map on $\hat{\mathbb{C}}$, denoted by f_ω; this map fixes the unit disc Δ and its boundary S^1 [4,16]. The map f_ω depends continuously on ω and satisfies

$$f_\omega(e^{i\theta}) = e^{iF_\omega(\theta)}. \quad (16)$$

So F_ω is a lift to $\mathbb{R}$ of the Möbius map f_ω restricted to the unit circle. A Möbius map of the disc to itself which is not the identity map can be classified into one of three types: elliptic, which is conjugate to a rotation of the disc; hyperbolic, which has an attracting-repelling pair of fixed points on the circle; and parabolic, which has a unique fixed point on the circle that is globally (but not locally) attracting on the circle [17]. We now state and prove our main result.

Theorem. (A) The average frequency function ρ for equation (5) is strictly increasing on each interval $[\omega_k^+, \omega_{k+1}^-]$; (B) we have $\omega_k^- = \omega_k^+$ if and only if f_ω is the identity map for some $\omega \in [\omega_k^-, \omega_k^+]$.

Proof. (A) The rotation number of the Möbius map f_ω restricted to the circle is the rotation number mod 2π of its lift F_ω and hence is $\rho(\omega)T$. mod 2π This rotation number is 0 mod 2π if and only if f_ω has a fixed point on the circle [3]. Therefore if $\omega \in (\omega_k^+, \omega_{k+1}^-)$ for some $k \in \mathbb{Z}$, then f_ω has no fixed points on the circle and so is elliptic, conjugate to a rotation on Δ of the form $\phi(z) = e^{i\alpha}z$. This implies that the lift F_ω is conjugate to a translation, and then our lemma implies that ρ is strictly increasing on $[\omega_k^+, \omega_{k+1}^-]$.

(B) Observe that if f_ω is the identity map for some $\omega \in [\omega_k^-, \omega_k^+]$, then its lift F_ω is translation by $2\pi k$ on $\mathbb{R}$; then our lemma implies that ρ is strictly increasing at ω, so we must have $\omega_k^- = \omega_k^+$. Conversely, suppose $\omega_k^- = \omega_k^+$. The map $f_{\omega_k^-}$ is a limit of elliptic maps and so must be either parabolic or the identity. If $f_{\omega_k^-}$ is parabolic, then f_ω has a unique fixed point $e^{i\theta_0}$ on S^1. The graph of the lift $y = F_{\omega_k^-}(\theta) - 2\pi k$ is tangent to the line $y = \theta$ at the fixed points at $\theta_0 + 2\pi n$, $n \in \mathbb{Z}$ and elsewhere lies completely either above or below the line $y = \theta$. We know that F_ω is strictly increasing in ω; therefore the family $F_\omega - 2\pi k$ has a tangent bifurcation at $\omega = \omega_k^-$, which implies that $F_\omega - 2\pi k$ has fixed points for a nontrivial closed interval containing ω_k^-, but this implies that $\omega_k^- < \omega_k^+$. Therefore $f_{\omega_k^-}$ is the identity map. ∎

Our theorem does not guarantee the existence of integer plateaus; in fact for the classic unimodal Kuramoto model there is only the one integer plateau corresponding to phase locking. However, based on the proof of our theorem, one would expect integer plateaus to exist for general Kuramoto oscillators, as they do in the case of the bimodal Kuramoto model. This is because in the three-dimensional group of Möbius maps that preserve the unit disc, the open set of elliptic maps has two-dimensional boundary consisting of the parabolic maps and the identity. So if $\omega_k^- = \omega_k^+$ (i.e., no plateau at $2\pi k/T$), then the curve $\omega \mapsto f_\omega$ in the space of Möbius maps touches the boundary of the elliptic region at the identity map when $\omega = \omega_k^-$ but remains inside the elliptic region for $\omega \neq \omega_k^-$ in some open interval containing ω_k^-. This is possible, but certainly atypical for a generic one-parameter family f_ω in this space. Thus we expect that in the general case of periodically forced Kuramoto oscillators (5), the average frequency function has plateaus at all (or at least most) integer multiples of the forcing frequency.

But the system (5) often has some symmetry or invariance causing some integer plateaus to degenerate to points, as occurs for the even multiples of Ω_{op} in the bimodal Kuramoto model in Fig. 1(b). This can be explained as follows: The Kuramoto model (1) has a one-parameter group of symmetries which preserve solutions, given by $\tilde{\theta}_j(t) = \theta_j(t) + c$ for any constant $c \in \mathbb{R}$. Consequently no solution to (1) can be stable in all directions. Here's one way to break this symmetry: suppose $\omega_{-j} = -\omega_j$ for all j (so $\omega_0 = 0$). Then the subset of the full state space defined by $\theta_{-j} = -\theta_j$ for all j is invariant under the flow for (1), and the reduced system is given by

$$\dot{\theta}_j = \omega_j - Kr(t)\sin\theta_j, \qquad j = 0, \ldots, N \tag{17}$$

with real-valued order parameter

$$r(t) = \frac{1}{2N+1}\left(\cos\theta_0 + 2\sum_{k=1}^{N}\cos\theta_k\right). \tag{18}$$

This reduced system still has a discrete symmetry given by $\tilde{\theta}_j(t) = \theta_j(t) + \pi$, which has corresponding order parameter $\tilde{r}(t) = -r(t)$. In the infinite-N limit, the bimodal Kuramoto system undergoes a Hopf bifurcation away from the incoherent solution with $r(t) = 0$ to a unique stable periodic solution [9], which therefore must be invariant under the symmetry $\theta \mapsto \theta + \pi$. Let $r(t)$ be the order parameter for this stable solution, with period $T > 0$. Since this symmetry reverses the order parameter, we see that $r(t)$ and $-r(t)$ must be time shifts of each other by phase $T/2$, i.e., $r(t + T/2) = -r(t)$ Now suppose that $\theta(t)$ is a solution to (17) for some ω; then using this, it's easy to verify that

$$\psi(t) = \theta(t + T/2) + \pi \tag{19}$$

is also a solution to (17) for this ω.

If the function ρ has a plateau at some value ω, then the Möbius transformation f_ω must be hyperbolic or parabolic, and therefore has a unique attracting fixed point on the circle. This means that for this ω, (17) has a unique attracting solution $\theta(t)$, up to multiples of 2π. But then we must have $\theta(t + T/2) + \pi = \theta(t) + 2\pi k$ for some $k \in \mathbb{Z}$, which implies that

$$\theta(t + T) = \theta(t) + (2k - 1)2\pi \tag{20}$$

and hence $\rho(\omega) = (2k-1)2\pi/T$ must be an odd multiple of Ω_{op}. Moreover, this argument applies whenever (5) is invariant under the symmetry (19).

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