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SPLAY-PHASE ORBITS FOR EQUIVARIANT FLOWS ON TORI*

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Abstract. This paper studies dynamical systems on the n-fold torus equivariant under a cyclic permutation of coordinates. It is proved that under a mild condition, these systems have splay-phase solutions. These are periodic orbits in which the n coordinates are given by the same function of time, but equally separated in phase. Applications to systems of equations used to model Josephson junction arrays are discussed.

Key words. Josephson junctions, ponies on a merry-go-round, splay-phase orbits

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This work is part of an ongoing effort to apply the techniques of nonlinear systems to understand the behavior of Josephson junction arrays. Josephson junction arrays are superconducting electronic devices, capable of generating very high frequency voltage oscillations, up to 10^{11} Hz or more. We refer the reader to [4], [5], and [7]–[14] for more information about Josephson junctions. To mathematicians, the most important feature of the system of equations governing Josephson junction arrays is the high degree of symmetry present in the system. We shall prove that certain types of solutions to these equations, which we call splay-phase solutions, exist under a mild condition. Moreover, the existence of these solutions essentially follows from nothing more than the symmetry present in the Josephson junction equations. Hence our results apply to any system of differential equations possessing this symmetry. Accordingly, we will present our results in as general a setting as possible.

The simplified equations we studied in [12] have the form

$$\theta_i' = c_1 + c_2 \sin\theta_i + c_3 \sum_{j=1}^{n} \sin\theta_j, \tag{1}$$

where $i = 1, \ldots, n, c_1, c_2, c_3$ are constants and $(\theta_1, \ldots, \theta_n)$ is a point on the n-fold torus T^n. This system is equivariant under any permutation of the coordinates θ_i (we will explain precisely what this means below). We studied two types of solutions to (1) in [12]: in-phase and splay-phase solutions. In-phase solutions are, of course, solutions $(\theta_1(t), \ldots, \theta_n(t))$, where $\theta_i(t) = \theta_j(t)$ for all t. Splay-phase solutions are solutions of the form

$$\left(\phi(t), \phi\left(t + \frac{1}{n}T\right), \ldots, \phi\left(t + \frac{n-1}{n}T\right)\right), \tag{2}$$

where ϕ has period exactly T. In other words, the coordinates each have the same periodic behavior, but are equally staggered in phase. (For simplicity, we order the phase shifts to correspond to the ordering of the coordinates, but note that solutions to (1) are preserved by any permutation of the coordinates.) Other authors call these "wagon wheel" or "ponies on a merry-go-round" solutions [4], [5].

Aronson, Golubitsky, and Mallet-Paret prove splay-phase solutions exist for this system in [5]. Their proof uses functional analysis methods, and applies more generally to a system like (1) with second derivative terms. We shall give a proof that (1) has

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splay-phase solutions using the Lefschetz trace formula. The only property of (1) that the proof relies on is that the system is equivariant under a cyclic permutation of coordinates.

Consider a system of differential equations on the torus T^n:

$$\theta_i' = f_i\left(\theta_1, \ldots, \theta_n\right). \tag{3}$$

Define σ: $T^n \to T^n$ by the rule $\sigma(\theta_1, \ldots, \theta_n) = (\theta_2, \ldots, \theta_n, \theta_1)$. Let F_t be the time t flow for (3). Then we say (3) is *σ-equivariant* if for all $t, F_t \circ \sigma = \sigma \circ F_t$. This means a curve $(\theta_1(t), \ldots, \theta_n(t))$ is an orbit for (2) if and only if $(\theta_2(t), \ldots, \theta_n(t), \theta_1(t))$ is. In terms of the components of the flow, σ-equivariance means

$$f_i\left(\theta_1, \ldots, \theta_n\right) = f_1\left(\theta_i, \ldots, \theta_n, \theta_1, \ldots, \theta_{i-1}\right) \quad \text{for } i = 2, \ldots, n. \tag{4}$$

The Josephson junction model (1) is an example of a σ-equivariant flow on T^n. Another example is a "ring of coupled oscillators" given by equations

$$\theta_i' = c_1 + c_2 \sin\left(\theta_{i+1} - \theta_i\right) + c_3 \sin\left(\theta_{i-1} - \theta_i\right), \tag{5}$$

where $i = 1, \ldots, n, c_1, c_2, c_3$ are constants and we interpret all indices mod n. (See [1]–[3] and [6] for a discussion of this example. Ermentrout found conditions for the stability of splay-phase solutions for this model in [6].) We now state and prove the theorem.

THEOREM. *Let $\theta_i' = f_i(\theta_1, \ldots, \theta_n)$ be a σ-equivariant flow on T^n. Suppose also that*

$$\sum_{i=1}^{n} f_i\left(\theta_1, \ldots, \theta_n\right) > 0 \quad \textit{for all } \left(\theta_1, \ldots, \theta_n\right) \in T^n. \tag{6}$$

Then this system has splay-phase orbits.

Remark 1. For the Josephson array (1), this condition is

$$nc_1 + (c_2 + nc_3) \sum_{j=1}^{n} \sin\theta_j > 0 \tag{7}$$

which is true for all $(\theta_1, \ldots, \theta_n) \in T^n$ exactly if $c_1 > |c_2/n + c_3|$. In the ring of oscillators (5), a necessary condition guaranteeing (6) is $c_1 > |c_2 - c_3|$. (This condition is sufficient when $n \equiv 0 \bmod 4$, or as $n \to \infty$.)

Remark 2. The Josephson junction model (1) can have splay-phase solutions even when condition (6) does not hold. See [12] for details.

Remark 3. Obviously a similar result holds if instead of (6) we assume

$$\sum_{i=1}^{n} f_i\left(\theta_1, \ldots, \theta_n\right) < 0 \quad \text{for all } \left(\theta_1, \ldots, \theta_n\right) \in T^n. \tag{8}$$

Remark 4. As mentioned above, the Josephson junction model (1) is of course equivariant under any permutation of the coordinates θ_i. This system is also invariant under time reversal, in the sense that $(\theta_1(t), \ldots, \theta_n(t))$ is an orbit if and only if $(\pi - \theta_1(-t), \ldots, \pi - \theta_n(-t))$ is. Our arguments do not rely on these symmetries (again, see [12] for more details).

Proof of the Theorem. Consider the $n-1$ torus $\Sigma \subset T^n$ given by $\sum_{i=1}^n \theta_i \equiv 0 \bmod 2\pi$. Since

$$\sum_{i=1}^{n} \theta_i' = \sum_{i=1}^{n} f_i(\theta_1, \dots, \theta_n) > 0, \tag{9}$$

there is a well-defined Poincaré first return map on Σ which we denote by $\Phi : \Sigma \to \Sigma$. Actually, this is the only place we use condition (6). We make two claims, which suffice to prove the theorem. □

Claim 1. Suppose $p \in \Sigma$ satisfies $\Phi(p) = \sigma(p)$. Then the orbit of p is a splay-phase solution.

Claim 2. The equation $\Phi(p) = \sigma(p)$ has solutions $p \in \Sigma$.

Proof of Claim 1. Suppose $\Phi(p) = \sigma(p)$ for some $p \in \Sigma$. Let $T/n > 0$ be the time required for p's first return to Σ. Recall that we denote the time t flow map $F_t : T^n \to T^n$. Now

$$F_{T/n}(p) = \Phi(p) = \sigma(p), \tag{10}$$

and since $F_t \circ \sigma = \sigma \circ F_t$ for all t,

$$F_T(p) = \left(F_{T/n}\right)^n(p) = \sigma^n(p) = p. \tag{11}$$

Hence the orbit of p is periodic, with period T/m for some integer $m > 0$. Let

$$(\theta_1(t), \dots, \theta_n(t)) = F_t(p),$$

$$\phi(t) = \theta_n(t). \tag{12}$$

Then since

$$F_{t+T/n}(p) = F_t(\sigma(p)) = \sigma(F_t(p)), \tag{13}$$

we see that

$$\theta_i(t) = \phi\left(t + \frac{T}{n} i\right), \qquad i = 1, \dots, n. \tag{14}$$

It remains to prove that $m = 1$. Now

$$\sum_{i=1}^{n} \theta_i'(t) = \sum_{i=1}^{n} \phi'\left(t + \frac{T}{n} i\right), \tag{15}$$

so

$$\int_0^{T/n} \sum_{i=1}^{n} \theta_i'(t)\, dt = \int_0^T \phi'(t)\, dt. \tag{16}$$

Now

$$\int_0^{T/n} \sum_{i=1}^{n} \theta_i'(t)\, dt = 2\pi, \tag{17}$$

since T/n is the first return time for p back to Σ. Hence

$$\int_0^T \phi'(t)\, dt = 2\pi. \tag{18}$$

But if ϕ has period T/m, then $\int_0^T \phi'(t)dt$ is a multiple of $2\pi m$. Hence $m = 1$. □

Proof of Claim 2. We need to prove that the map $\sigma^{-1} \circ \Phi$ has a fixed point on Σ. We shall do this by applying the Lefschetz fixed point theorem to the map $\sigma^{-1} \circ \Phi$. First we show that the map $\Phi : \Sigma \to \Sigma$ is homotopic to the identity map. We exhibit the homotopy as follows. For any $p \in \Sigma$, let $t(p)$ be the time required for p to return to Σ. For $0 \le s \le 1$, let

$$(\theta_1(p,s), \ldots, \theta_n(p,s)) = F_{s\cdot t(p)}(p). \tag{19}$$

Then set

$$G_s(p) = (\theta_1(p,s), \ldots, \theta_{n-1}(p,s), -(\theta_1(p,s) + \cdots + \theta_{n-1}(p,s))). \tag{20}$$

For $s = 0, G_0(p) = p$, and for $s = 1, G_1(p) = F_{t(p)}(p) = \Phi(p)$, so G_s is the required homotopy.

Finally, we recall the Lefschetz theorem. If $f : X \to X$ is a continuous map on, for example, a compact n-dimensional manifold X, then the Lefschetz number of f is given by

$$\lambda(f) = \sum_{p=0}^{n} (-1)^p \operatorname{Tr}(f_p, H_p(X, \mathbf{Q})), \tag{21}$$

where f_p is the induced map on the homology group $H_p(X, \mathbf{Q})$, and Tr denotes trace. Of course, $\lambda(f)$ depends only on the homotopy class of f. Lefschetz's theorem says that if $\lambda(f) \ne 0$, then f has at least one fixed point. Under additional assumptions, one may also calculate the Lefschetz number of a map as the sum of local contributions. Suppose that f is a smooth map, with finitely many fixed points $p_1, \ldots, p_m$, all nondegenerate. Then

$$\lambda(f) = \sum_{k=1}^{m} i_{p_k}(f), \tag{22}$$

where the local index $i_{p_k}(f)$ is given by

$$i_{p_k}(f) = \operatorname{sgn} \det (I - df_{p_k}), \tag{23}$$

where df_{p_k} is the derivative of f on the tangent space $T_{p_k}X$.

In our situation, we wish to prove that the map $\sigma^{-1} \circ \Phi$ on Σ has a fixed point. Since Φ is homotopic to the identity map, $\sigma^{-1} \circ \Phi$ is homotopic to σ^{-1}, so it suffices to show that $\lambda(\sigma^{-1}) \ne 0$. And since $\lambda(\sigma) = \lambda(\sigma^{-1})$, we might as well work on σ. So σ has n fixed points on Σ, given by

$$\left(\frac{2\pi k}{n}, \ldots, \frac{2\pi k}{n}\right) \quad \text{for } k = 0, 1, \ldots, n-1. \tag{24}$$

They all have the same local structure, so it suffices to calculate the local index at $(0, \ldots, 0)$. In local coordinates $(\alpha_1, \ldots, \alpha_{n-1}) \to (\alpha_1, \ldots, \alpha_{n-1}, -(\alpha_1 + \cdots + \alpha_{n-1}))$ near $(0, \ldots, 0)$ on Σ, σ is given by the linear map

$$A(\alpha_1, \ldots, \alpha_{n-1}) = (\alpha_2, \ldots, \alpha_{n-1}, -(\alpha_1 + \cdots + \alpha_{n-1})). \tag{25}$$

The characteristic equation of this linear map is

$$\det(\mu I - A) = \frac{\mu^n - 1}{\mu - 1} = 1 + \mu + \cdots + \mu^{n-1}. \tag{26}$$

Hence $\det(I-A) = n$, so $\operatorname{sgn}\det(I-A) = 1$. Therefore $\lambda(\sigma) = n$, so we are done. In fact, it suffices to observe that $\det(I-A) \neq 0$ since all the fixed points of σ on Σ have the same local type. And $\det(I-A) \neq 0$ because the equation $A\mathbf{x} = \mathbf{x}, \mathbf{x} \in \mathbf{R}^{\mathbf{n-1}}$ has no nonzero solutions. □

Concluding remarks. The next question to ask about splay-phase solutions is whether they are stable. Based on computer simulation, we believe that the splay-phase orbits in the Josephson junction model (1) are neutrally stable. We can only prove this analytically when $n = 2$ (see [12]). In general, this question remains open. However, Strogatz and the author have made some progress towards understanding the stability of splay states. We have analyzed an infinite-dimensional analog of the Josephson junction system, and calculated the Floquet multipliers around the splay states. Our calculations predict neutral stability for the system (1) considered in this paper. We conjecture that the splay states in the finite-dimensional Josephson junction system (1) have the same stability behavior as in our infinite-dimensional model. See [11].

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