Splay states in globally coupled Josephson arrays: Analytical prediction of Floquet multipliers

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In recent numerical experiments on series arrays of overdamped Josephson junctions, Nichols and Wiesenfeld [Phys. Rev. A **45**, 8430 (1992)] discovered that the periodic states known as splay states are neutrally stable in all but four directions in phase space. We present a theory that accounts for this enormous degree of neutral stability. The theory also predicts the four non-neutral Floquet multipliers to within 0.1% of their numerically computed values. The analytical approach used here may be applicable to other globally coupled systems of oscillators, such as multimode lasers, electronic oscillator circuits, and solid-state laser arrays.

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I. INTRODUCTION

This paper was inspired by the puzzling results of some recent numerical experiments on series arrays of overdamped Josephson junctions. While investigating the stability of certain collective oscillations known as splay states, Nichols and Wiesenfeld [1] discovered that these states are *neutrally stable to perturbations in all but four directions* in phase space. Moreover, this huge degree of neutral stability was found to persist over a wide range of parameters and for a variety of different loads. The robustness of the phenomenon is especially surprising, given that Josephson arrays are dissipative dynamical systems; normally such systems exhibit neutral stability only at isolated parameter values. The goal of this paper is to present a theory that accounts for the observations of Nichols and Wiesenfeld [1].

The problem considered here is part of a larger effort to apply dynamical systems theory to the analysis of Josephson arrays [1–10]. Following previous authors, we restrict attention to the most symmetric possible case: a series array of identical junctions in which each junction is coupled equally to all the others. At first this "global coupling" may seem artificial, but it actually follows quite naturally from Kirchhoff's laws. In fact, global coupling also arises in several other physical systems, including multimode lasers with an intracavity crystal [11–13], electronic oscillator circuits [14], and laser arrays [15].

Each of these systems may be regarded as a population of coupled nonlinear oscillators. Such systems can exhibit a variety of collective modes, the simplest of which is the coherent mode where all the oscillations are in phase. Early studies [2,3] focused on the stability of this mode because of its importance for technological applications [16].

More recently, attention has turned to a different kind of periodic state, variously known as the antiphase state [2], rotating wave [14], ponies on a merry-go-round [8,9], or splay state [5]. The term "splay state" is motivated by the state's appearance when plotted on a phasor diagram: The phases of the oscillators are splayed apart on the unit circle. To be more precise, the defining property of a splay state is that all the oscillations have the same wave form, but are staggered equally in time. Thus if $x_k(t)$ is the state of the kth oscillator, then $x_k(t)=X(t+kT/N)$, for $k=1,\ldots,N$, where X is the common wave form, T is the oscillation period, and N is the number of oscillators.

Splay states were first observed by Hadley and Beasley [2] in numerical simulations of Josephson arrays. They have since been detected experimentally in a multimode laser system [12] and in an electronic oscillator circuit [14]. Their existence for all N has been proven rigorously for two particular Josephson arrays [9], and in a model system of symmetrically coupled phase oscillators [17].

The significance of splay states is that they occur in vast numbers, if they occur at all; if one splay state exists, then in fact there are $(N-1)!$ of them, obtained by permuting the indices of the oscillators. This explosive proliferation can lead to attractor crowding [18], a phenomenon in which the array becomes increasingly sensitive to noise as N grows. On the other hand, the multiplicity of splay states raises the technologically intriguing possibility that they might be used as storage elements in a dynamic, rewritable memory [19]—assuming, of course, that the splay states were stable, and that one could switch reliably among them.

So the natural question becomes: Under what circumstances are splay states stable? In the first theoretical attack on this question, Tsang *et al.* [5] and Swift, Strogatz, and Wiesenfeld [10] studied the simple case of Josephson arrays with overdamped (zero capacitance) junctions and a pure resistive load. To their surprise, they found that the splay states are neutrally stable in all directions: all the Floquet multipliers [20] lie on the unit circle. Tsang *et al.* [5] originally guessed that this highly nongeneric property might be related to the system's time-reversal symmetry, but then Tsang and Schwartz [7] gave an example of a nonreversible Josephson array which nonetheless had neutrally stable splay states. The neutral stability was not quite as severe as in the reversible case, but was still peculiarly large: Tsang and Schwartz [7] reported that all but four of the Floquet multipliers equal $+1$ "up to single precision machine accuracy."

Now Nichols and Wiesenfeld [1] have extended this result to a much broader class of arrays. In particular, for arrays of overdamped junctions with various *RLC* loads, they once again find that all but four of the Floquet multipliers equal $+1$. They also report careful numerical estimates of the four non-neutral Floquet multipliers.

In this paper we present a theory that explains the origin of the neutral stability. The culprit turns out to be the pure sinusoidal form of the Josephson current relation [21]. The lack of higher harmonics leads to a massive decoupling of the governing dynamical equations, at least in the neighborhood of the splay state. This effect has been noticed previously in three model systems [22–24], and is now seen to occur in realistic models of Josephson arrays as well.

The theory also quantitatively predicts the values of the four non-neutral Floquet multipliers. Though based on the infinite-N limit, the theory is accurate for small N. For instance, the predicted Floquet multipliers agree with the numerically computed values to less than 0.1%, even when $N=4$.

II. GOVERNING EQUATIONS

We consider the class of arrays shown in Fig. 1. An *RLC* load is in parallel with a series array of identical Josephson junctions. We assume that the junctions are heavily overdamped, i.e., they have negligible capacitance. (The case of nonzero junction capacitance remains unsolved. It is more difficult and requires additional techniques, as we discuss in Sec. V.)

The dynamics of the array are given by Kirchhoff's laws and the Josephson relations [2,21]. The equations are

$$\frac{\hbar}{2er}\frac{d\phi_k}{dt}+I_c\sin\phi_k+\frac{dQ}{dt}=I_b,\quad k=1,\ldots,N$$

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=\frac{\hbar}{2e}\sum_{j=1}^{N}\frac{d\phi_j}{dt}\ ,$$

from Kirchhoff's current law and voltage law, respectively. Here $\hbar$ is Planck's constant divided by 2π, e is the electron charge, r is the junction resistance, $\phi_k(t)$ is the phase difference across junction k, I_c is the critical current, $Q(t)$ is the charge on the capacitor, I_b is the dc-bias current, and L, R, and C are the inductance, resistance, and capacitance of the load, respectively. Note the permutation symmetry of the equations, as well as the presence of global coupling in the second equation.

Now we nondimensionalize the system in a way that yields a nontrivial limit as $N\to\infty$. This scaling [2,3] differs from the conventional scaling by a factor of N in the dimensionless versions of L, R, and C. In what follows, the quantities with an asterisk are dimensionless. Let

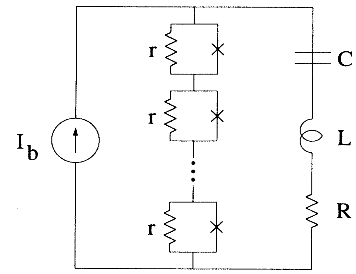

FIG. 1. Circuit schematic for a series array of identical overdamped Josephson junctions with an *RLC* load and a dc-current bias.

$$\omega_c=2erI_c/\hbar,\quad t^*=\omega_c t,\quad Q^*=\omega_c Q/I_c,\quad I_b^*=I_b/I_c\ ,$$

$$L^*=\frac{\omega_c L}{rN},\quad R^*=\frac{R}{rN},\quad C^*=N\omega_c rC\ .$$

After substituting these expressions into the governing equations and dropping the asterisks, we obtain

$$\dot\phi_k+\sin\phi_k+\dot Q=I_b,\quad k=1,\ldots,N, \tag{1}$$

$$L\ddot Q+R\dot Q+C^{-1}Q=\frac{1}{N}\sum_{j=1}^{N}\dot\phi_j\ , \tag{2}$$

where the overdot denotes differentiation with respect to dimensionless time.

We want to study the dynamical system (1),(2) in the large-N limit. The appropriate infinite-dimensional system is suggested by an analogy with fluid mechanics. As $N\to\infty$, imagine a continuum of oscillators flowing around the unit circle. The velocity field on the circle is given by

$$v(\phi,t)=I_b-\dot Q-\sin\phi\ ,$$

from (1). Let $\rho(\phi,t)$ denote the density of oscillators at phase ϕ at time t. Then we can replace the average in (2) by an integral:

$$\frac{1}{N}\sum_{j=1}^{N}\dot\phi_j\to\int_0^{2\pi}\dot\phi\rho(\phi,t)d\phi$$

$$=\int_0^{2\pi}[I_b-\dot Q-\sin\phi]\rho(\phi,t)d\phi\ .$$

Hence (2) becomes

$$L\ddot Q+(R+1)\dot Q+C^{-1}Q=I_b-\int_0^{2\pi}\rho(\phi,t)\sin\phi\,d\phi\ . \tag{3}$$

The density $\rho(\phi,t)$ evolves according to the continuity equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial\phi}(\rho v)=0,$$

which expresses "conservation of oscillators." Substituting for $v(\phi,t)$ yields

$$\frac{\partial\rho}{\partial t}=-\frac{\partial}{\partial\phi}(\rho[I_b-\dot{Q}-\sin\phi]). \tag{4}$$

Together (3) and (4) determine the evolution of the infinite-N system, which has state variables Q, $\dot{Q}$, and ρ.

III. SPLAY STATE: EXISTENCE

In the infinite-N limit, a splay state corresponds to a *stationary* solution of (3) and (4). This makes sense intuitively—in the splay state, all the oscillators are equally staggered in time and move around the circle in exactly the same way. Therefore, at a macroscopic level, the flow around the circle is steady and ρ is independent of time.

The stationarity of the splay state implies that $\partial\rho/\partial t=0$, $\dot{Q}=0$, and $\ddot{Q}=0$. Note that by requiring $\dot{Q}=0$, we are assuming that there actually *is* a capacitor in the load, i.e., $C\neq\infty$, in which case no current passes through the load in steady state. If instead there were no capacitor, the stationary solution would have I=const, where I is the current passing through the load. The results in this case will be discussed at the end of Sec. IV.

The density for the stationary solution is given by

$$\rho=\rho_0(\phi)=\frac{v}{I_b-\sin\phi},$$

where the normalization constant

$$v=(2\pi)^{-1}(I_b^2-1)^{1/2}$$

is determined by $\int_0^{2\pi}\rho_0(\phi)d\phi=1$. (Here and from now on, we assume that $I_b>1$; otherwise the oscillators get stuck at some point on the circle and the splay state does not exist.) The constant charge on the capacitor is $Q=Q_0$, where Q_0 satisfies

$$C^{-1}Q_0=I_b-\int_0^{2\pi}\rho_0(\phi)\sin\phi\,d\phi,$$

from (3). After substituting for ρ_0 and evaluating the integral, we find $Q_0=C(I_b^2-1)^{1/2}$.

IV. SPLAY STATE: LINEAR STABILITY

This section contains the main part of the analysis. In Sec. IV A we linearize (3),(4) about the stationary density ρ_0. Then in Sec. IV B we find the eigenvalues of the linearized system. We claim that these eigenvalues are intimately related to the Floquet exponents and multipliers [20] about the splay state of (1),(2). Unfortunately we have not been able to prove a rigorous result in this direction, so we merely state the idea as a conjecture.

Conjecture. In the limit $N\to\infty$, the Floquet exponents about the splay state of (1),(2) converge to $\{\lambda_i\}$. Here λ_i is an eigenvalue of the infinite-dimensional system (3),(4) linearized about the stationary density ρ_0. The corresponding Floquet multipliers converge to $\{\exp(\lambda_i T)\}$, where $T=2\pi(I_b^2-1)^{-1/2}$ is the period of the splay state as $N\to\infty$.

The intuition here is that the Floquet exponents for the finite-dimensional problem play the same role as the eigenvalues for the infinite-dimensional problem; both govern the evolution of infinitesimal perturbations about the splay state. In any case, the numerical evidence for the conjecture is compelling, as shown in Sec. IV C.

A. Linearized system

First we consider the linear stability of the stationary state $(Q,\dot{Q},\rho)=(Q_0,0,\rho_0)$. Let $Q=Q_0+q$, $\dot{Q}=\dot{q}$, $\rho=\rho_0+\eta$. Then the linearization of (3),(4) is

$$L\ddot{q}+(R+1)\dot{q}+C^{-1}q=-\int_0^{2\pi}\eta(\phi,t)\sin\phi\,d\phi, \tag{5}$$

$$\frac{\partial\eta}{\partial t}=-\frac{\partial}{\partial\phi}(\eta\nu_0)+\dot{q}\frac{d\rho_0}{d\phi}, \tag{6}$$

where

$$\nu_0(\phi)=I_b-\sin\phi$$

is the velocity corresponding to the splay state. To analyze this linear system, a natural strategy would be to express $\eta(\phi,t)$ as a Fourier series in ϕ, and then derive a set of amplitude equations. Golomb *et al.* [23] recently did such a calculation for a closely related system. They introduced two clever changes of variables that make the calculation easier. In what follows, we adopt the notation of Appendix B in Golomb *et al.* [23].

The first trick is to reparametrize the circle to remove the angular dependence in the velocity $\nu_0(\phi)$. Let

$$G(\phi)=\int_0^{\phi}\rho_0(\phi')d\phi'.$$

This is equivalent to the transformation to "natural angles" in Swift, Strogatz, and Wiesenfeld [10]. As ϕ runs from 0 to 2π, G runs from 0 to 1. Thus $\theta=2\pi G(\phi)$ may be regarded as a new angular variable. It is "natural" in the following sense: When the system is in the splay state, each oscillator moves at *constant* angular velocity with respect to θ, because

$$\begin{aligned}\dot{\theta}&=2\pi G'(\phi)\dot{\phi}\\&=2\pi\rho_0\nu_0\\&=2\pi v\\&=(I_b^2-1)^{1/2}.\end{aligned}$$

Furthermore, in this coordinate system the splay state corresponds to a *uniform* distribution of oscillators around the circle.

The second trick is to express $\eta(\phi,t)$ as follows:

$$\eta(\phi,t)=\rho_0(\phi)\sum_{m=-\infty}^{\infty}a_m(t)e^{2\pi imG(\phi)}.$$

Note the prefactor ρ_0; it will turn out to be useful. Also, the Fourier series is with respect to the natural angle $2\pi G(\phi)$, rather than ϕ itself. At $t=0$, the amplitudes a_m satisfy $\bar{a}_m=a_{-m}$, by reality of η, and $a_0=0$, by the normalization of ρ; we will see later that these conditions hold for *all t*.

In terms of the new variables, Eq. (5) becomes

$$L\ddot{q}+(R+1)\dot{q}+C^{-1}q$$
$$=-\int_0^{2\pi}\rho_0(\phi)\sum_{m=-\infty}^{\infty}a_m(t)e^{2\pi imG(\phi)}\sin\phi\,d\phi\ .$$

This may be written compactly as

$$L\ddot{q}+(R+1)\dot{q}+C^{-1}q=-\sum_{m=-\infty}^{\infty}f_m a_m\ , \tag{7}$$

where f_m is defined by

$$f_m=\int_0^{2\pi}\rho_0(\phi)e^{2\pi imG(\phi)}\sin\phi\,d\phi\ .$$

Our f_m is a special case of that defined in Golomb *et al.* [23], Eq. (B7); here $g(\phi)=\sin\phi$ in their notation.

Next we rewrite (6). The advantage of the new variables will now become obvious. We find

$$\rho_0\sum_{m=-\infty}^{\infty}\dot{a}_m e^{2\pi imG(\phi)}=\dot{q}\frac{d\rho_0}{d\phi}-\frac{\partial}{\partial\phi}\left[v\sum_{m=-\infty}^{\infty}a_m e^{2\pi imG(\phi)}\right]$$

since $\eta\nu_0=\rho_0\nu_0\sum_m a_m e^{2\pi imG(\phi)}$, and $\rho_0\nu_0=v$. Carrying out the differentiation on the right-hand side and recalling that $dG/d\phi=\rho_0$, we obtain

$$\rho_0\sum_{m=-\infty}^{\infty}\dot{a}_m e^{2\pi imG(\phi)}=\dot{q}\frac{d\rho_0}{d\phi}-v\rho_0 \times\sum_{m=-\infty}^{\infty}2\pi ima_m e^{2\pi imG(\phi)}\ .$$

To find the amplitude equations, we multiply both sides by $e^{-2\pi inG(\phi)}$ and integrate from 0 to 2π, using the orthogonality relation

$$\int_0^{2\pi}e^{2\pi ikG(\phi)}\rho_0(\phi)d\phi=\begin{cases}0, & k\neq 0\\ 1, & k=0\end{cases}$$

[which follows from the usual relation $\int_0^{2\pi}e^{ik\theta}d\theta=2\pi\delta_{k,0}$ after the substitution $\theta=2\pi G(\phi)$]. The resulting amplitude equations are

$$\dot{a}_n=\dot{q}b_n-2\pi ivna_n\ , \tag{8}$$

where b_n is defined by

$$b_n=\int_0^{2\pi}\frac{d\rho_0}{d\phi}e^{-2\pi inG(\phi)}d\phi\ ,$$

as in Golomb *et al.* [23], Eq. (B6).

B. Eigenvalues

Now we study the eigenvalues of (7),(8). The problem is analytically tractable, because of the identity

$$b_n\equiv 0,\quad \text{for all } |n|\neq 1\ ,$$

as observed by Golomb *et al.* [23]. Thus Eqs. (7) and (8) reduce to

$$\dot{a}_n=-2\pi ivna_n,\quad n\neq\pm1, \tag{9a}$$

$$\dot{a}_1=b_1\dot{q}-2\pi iva_1\ , \tag{9b}$$

$$\dot{a}_{-1}=b_{-1}\dot{q}+2\pi iva_{-1}\ , \tag{9c}$$

$$L\ddot{q}+(R+1)\dot{q}+C^{-1}q=-\sum_{m=-\infty}^{\infty}f_m a_m\ . \tag{9d}$$

We make the following observations about (9).

(i) Equation (9a) shows that the evolution of the higher harmonics a_n, $|n|>1$, is independent of the rest of the system. Strogatz and Mirollo [22] and Golomb *et al.* [23] found a similar *decoupling of higher harmonics* for two other systems of oscillators. In each case, the decoupling stems from the pure sinusoidal form of the nonlinearity in the original system. Here, the identity $b_n\equiv 0$ for all $|n|\neq 1$ can ultimately be traced back to the $\sin\phi$ terms in Eqs. (3),(4).

(ii) Equation (9b) and (9c) are conjugates, since $\bar{b}_1=b_{-1}$ and $\bar{a}_1=a_{-1}$.

(iii) $a_0(t)\equiv 0$ since $\dot{a}_0=0$ and $a_0(0)=0$, as mentioned earlier.

To find the eigenvalues and eigenvectors of (9), we seek solutions of the form $a_n(t)=a_n(0)e^{\lambda t}$, $q(t)=q(0)e^{\lambda t}$. Then (9) becomes

$$\begin{aligned}&\lambda a_n=-2\pi ivna_n,\quad n\neq\pm1\\ &\lambda a_1=\lambda b_1 q-2\pi iva_1\ ,\\ &\lambda a_{-1}=\lambda b_{-1}q+2\pi iva_{-1}\ ,\\ &\lambda^2Lq+\lambda(R+1)q+C^{-1}q=-\sum_{m=-\infty}^{\infty}f_m a_m\ ,\end{aligned} \tag{10}$$

where now q,a_n denote initial values of those variables. There are two cases to consider.

Case 1: Pure imaginary eigenvalues

Because of the decoupling of the higher harmonics, one kind of eigenvector can be seen immediately. For any fixed $|k|>1$, pick $a_k=1$, and $a_n=0$ for all $n\neq\pm1$ and $n\neq k$. Then all the higher-harmonic equations in (10) are satisfied automatically, except for the case $n=k$, which yields

$$\lambda=-2\pi ivk\ .$$

To check that this is actually an eigenvalue, we need to verify that the remaining equations in (10) can be solved uniquely, given this value of λ. Those remaining equations are

$$\lambda a_1=\lambda b_1 q-2\pi iva_1\ ,$$

$$\lambda a_{-1}=\lambda b_{-1}q+2\pi iva_{-1},$$

$$\lambda^2Lq+\lambda(R+1)q+C^{-1}q=-[f_1a_1+f_{-1}a_{-1}+f_k]\ .$$

We have three linear equations in three unknowns: a_1, a_{-1}, and q. Generically, this system will have a unique solution. The only exception would be if the determinant of the system above were zero; this would require a certain resonance condition involving L, R, C, and I_b. Except at such special parameter values, the system has a unique solution, and so $\lambda=-2\pi i v k$ is indeed an eigenvalue for each integer $|k|>1$.

Hence we have an infinite number of distinct, pure imaginary eigenvalues. These eigenvalues account for the enormous neutral stability observed numerically by Nichols and Wiesenfeld. To see why, recall our conjecture that each λ corresponds to a Floquet multiplier $\mu=e^{\lambda T}$. Here $T=2\pi(I_b^2-1)^{-1/2}=1/v$ is the period of the splay state. Thus we obtain a simple formula for the Floquet multipliers:

$$\mu=e^{\lambda/v} .$$

But when $\lambda=-2\pi i v k$, this reduces to $\mu=e^{-2\pi i k}=1$. Thus the neutral stability may be traced back to the decoupling of the higher harmonics.

Case 2: Nontrivial eigenvalues

Now suppose that $a_n=0$ for all $n\neq\pm1$; in other words, the perturbation does not involve any higher harmonics. Then (10) becomes

$$\lambda a_1=\lambda b_1 q-2\pi i v a_1 , \tag{11a}$$

$$\lambda a_{-1}=\lambda b_{-1} q+2\pi i v a_{-1} , \tag{11b}$$

$$\lambda^2 L q+\lambda(R+1)q+C^{-1}q=-[f_1 a_1+f_{-1}a_{-1}] . \tag{11c}$$

To derive the characteristic equation for λ, we convert (11) to a system of equations with purely real coefficients. Let

$$a_1=A+iB,\quad a_{-1}=A-iB ,$$

$$b_1=K+iD,\quad b_{-1}=K-iD ,$$

$$f_1=\gamma+i\delta,\quad f_{-1}=\gamma-i\delta .$$

(Recall that the constants $b_{\pm1}$ and $f_{\pm1}$ have been defined earlier in terms of certain integrals; they will be evaluated explicitly later.) Adding and subtracting Eqs. (11a) and (11b) yields

$$\lambda A=\lambda K q+2\pi v B ,$$

$$\lambda B=\lambda D q-2\pi v A .$$

Next we break the second-degree equation (11c) into two first-degree equations by introducing a new variable

$$I=\lambda q$$

in terms of which (11c) becomes

$$\lambda L I+(R+1)I+C^{-1}q=-2[\gamma A-\delta B] .$$

In matrix form, these four equations imply

$$\begin{bmatrix} \lambda & -2\pi v & -\lambda K & 0 \\ 2\pi v & \lambda & -\lambda D & 0 \\ 0 & 0 & \lambda & -1 \\ 2\gamma & -2\delta & C^{-1} & \lambda L+R+1 \end{bmatrix}\begin{bmatrix} A \\ B \\ q \\ I \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} . \tag{12}$$

The eigenvalues satisfy the characteristic equation $\det(\mathbf{M})=0$, where $\mathbf{M}$ denotes the matrix in (12).

Before computing this determinant, we write down the explicit values of the constants K, D, δ, γ, using the identities in Golomb *et al.* [23], Appendix B. Equation (B15) in Golomb *et al.* [23] gives

$$b_1=\frac{e^{-2\pi i\hat{G}}}{2(I_b^2-1)^{1/2}},$$

where

$$2\pi\hat{G}=\tan^{-1}[1/(I_b^2-1)^{1/2}] .$$

(In translating from Golomb *et al.* [23], we have replaced their $\tilde{\omega}$ and A with I_b and -1, respectively.) Hence

$$e^{-2\pi i\hat{G}}=\frac{(I_b^2-1)^{1/2}-i}{I_b}$$

and therefore

$$K=\frac{1}{2I_b},\quad D=\frac{-1}{2I_b(I_b^2-1)^{1/2}} .$$

To find γ,δ, we use Eq. (B19) in Golomb *et al.* [23]:

$$f_1=ie^{2\pi i\hat{G}}(I_b^2-1)^{1/2}[I_b-(I_b^2-1)^{1/2}] .$$

After simplification this yields

$$\gamma=-\frac{\omega(I_b-\omega)}{I_b},\quad \delta=\frac{\omega^2(I_b-\omega)}{I_b} ,$$

where for notational convenience we have introduced the quantity

$$\omega=(I_b^2-1)^{1/2} ,$$

which is the angular frequency of the splay state.

Finally, after substituting the expressions for K, D, δ, and γ into (12) and using Mathematica to compute the determinant, we obtain the following equation for the eigenvalues:

$$L\lambda^4+(R+1)\lambda^3+(C^{-1}+L\omega^2)\lambda^2 +(I_b\omega+R\omega^2)\lambda+\omega^2C^{-1}=0. \tag{13}$$

Equation (13) is one of the main results of this paper. It explains why there are precisely four nontrivial Floquet multipliers. [Of course, if $L=0$, then (13) predicts there will be only three nontrivial multipliers.]

C. Numerical illustration

For particular choices of the parameters, we can solve (13) numerically and then compare the predicted multipliers $\mu=e^{\lambda/v}$ with those reported by Nichols and Wiesenfeld [1]. Actually, Nichols and Wiesenfeld reported the *magnitude* of the multipliers; our theory predicts

$$|\mu|=\left|\exp\left[\frac{\text{Re}\lambda+i\,\text{Im}\lambda}{v}\right]\right|=\exp\left[\frac{\text{Re}\lambda}{v}\right] .$$

Also, Nichols and Wiesenfeld [1] used a slightly different scaling from ours; they did not include any factors of N in the load parameters. To convert their dimensionless parameters to ours, let $\tilde{L}$, $\tilde{R}$, $\tilde{C}$ denote the parameters used by Nichols and Wiesenfeld [1]. Then

$$R=\tilde{R}/N,\quad L=\tilde{L}/N,\quad C=N\tilde{C}\ .$$

For example, consider Nichols and Wiesenfeld [1], case I, with $N=4$, $I_b=1.9$, $\tilde{R}=0$, $\tilde{L}=2$, $\tilde{C}=0.125$. In our notation,

$$N=4,\quad I_b=1.9,\quad R=0,\quad L=0.5,\quad C=0.5.$$

Then (13) becomes

$$0.5\lambda^4+\lambda^3+3.305\lambda^2+3.069\,54\lambda+5.22=0.$$

The roots are

$$\lambda_{1,2}=-0.964\,259\ldots\pm(1.576\,59\ldots)i\ ,$$

$$\lambda_{3,4}=-0.035\,741\,1\ldots\pm(1.747\,98\ldots)i\ .$$

Since $v=(2\pi)^{-1}(I_b^2-1)^{1/2}=0.257\,123\ldots$, the predicted multipliers have magnitudes

$$|\mu_{1,2}|=\exp\left[\frac{-0.964\,259\ldots}{0.257\,123\ldots}\right]=0.023\,513\,4\ldots\ ,$$

$$|\mu_{3,4}|=\exp\left[\frac{-0.035\,741\,1\ldots}{0.257\,123\ldots}\right]=0.870\,225\ldots\ .$$

For comparison, the values measured by Nichols and Wiesenfeld [1] are $|\mu_{1,2}|=0.023\,514\,9\ldots$ and $|\mu_{3,4}|=0.870\,140\ldots$. Hence there is agreement to at least three significant figures. For $N=10$, the agreement improves to at least four significant figures. This level of agreement is typical of all cases we have examined.

D. Stability criterion

By applying the Routh-Hurwitz criteria [25] to (13), we can obtain a necessary and sufficient condition for the nontrivial eigenvalues to lie in the left half plane $\mathrm{Re}\lambda<0$. If this condition is violated, the splay state is not merely neutrally stable, but genuinely *unstable*. Then attractor crowding [18] among the $(N-1)!$ distinct splay states would no longer be an issue, since they would no longer be even neutrally stable. On the other hand, such instability would eliminate the technologically interesting possibilities [19] of using such an array as a dynamic memory or "multiswitch."

According to the Routh-Hurwitz criteria [25], all the roots of the fourth-degree polynomial $\alpha_4\lambda^4+\alpha_3\lambda^3+\alpha_2\lambda^2+\alpha_1\lambda+\alpha_0$ lie in the left half plane if and only if the following quantities Δ_i are all positive:

$$\Delta_0=\alpha_4,\quad \Delta_1=\alpha_3,\quad \Delta_2=\alpha_3\alpha_2-\alpha_4\alpha_1\ ,$$

$$\Delta_3=\alpha_3\alpha_2\alpha_1-\alpha_0\alpha_3^2-\alpha_4\alpha_1^2\ ,$$

$$\Delta_4=\alpha_3(\alpha_2\alpha_1\alpha_0-\alpha_3\alpha_0^2)-\alpha_1^2\alpha_0\alpha_4.$$

From (13), we have

$$\alpha_4=L,\quad \alpha_3=R+1,\quad \alpha_2=C^{-1}+L\omega^2,$$

$$\alpha_1=I_b\omega+R\omega^2,\quad \alpha_0=\omega^2C^{-1}\ .$$

Thus $\Delta_0>0$ and $\Delta_1>0$ automatically. The condition $\Delta_2>0$ yields

$$I_b<\frac{R+1}{\omega LC}+\omega\ .$$

The conditions $\Delta_3>0$ and $\Delta_4>0$ turn out to be the same: They are both equivalent to

$$I_b<\frac{R+1}{\omega LC}-R\omega\ . \tag{14}$$

Notice that this condition is *stricter* than the condition $\Delta_2>0$, so it alone is the necessary and sufficient condition we seek. Hence all four nontrivial λ's satisfy $\mathrm{Re}\lambda<0$ *if and only if* (14) holds.

To clarify the implications of (14), consider the graphical analysis shown in Fig. 2. By definition of ω, the bias current is constrained to lie on the operating curve $I_b=(1+\omega^2)^{1/2}$. Let

$$\mu=(R+1)/LC.$$

Then according to (14), the splay state is neutrally stable if I_b lies below $\mu/\omega-R\omega$, and is unstable otherwise. Hence the splay state becomes unstable when I_b exceeds a threshold current I_{thresh}, defined by the intersection of the curves in Fig. 2. (On the other hand, I_b cannot be *too* small, since we require $I_b>1$ just for a splay state to *exist.)*

To find I_{thresh} explicitly, square both sides of $\mu/\omega-R\omega=(1+\omega^2)^{1/2}$, solve the resulting quadratic equation for ω^2, and choose the appropriate root. The result is

$$I_{\mathrm{thresh}}=\left[1+\frac{1+2R\mu-(1+4R\mu+4\mu^2)^{1/2}}{2(R^2-1)}\right]^{1/2}. \tag{15}$$

The negative square root inside is the right choice, because then I_{thresh} remains finite as $R\to1$, in accordance with Fig. 2; the other root blows up as $R\to1$, and is of no physical significance.

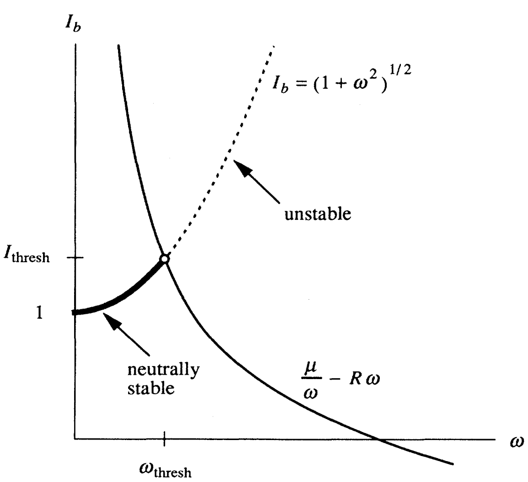

FIG. 2. Graphical analysis of stability condition (14). The splay state is neutrally stable for $1<I_b<I_{\mathrm{thresh}}$ (thick line) and unstable for $I_b>I_{\mathrm{thresh}}$ (dotted line).

Equation (15) is a bit opaque; it is easier to draw conclusions from Fig. 2. For instance, suppose we want to raise the threshold of instability. Then the curve $\mu/\omega - R\omega$ in Fig. 2 must move up. This can be achieved in various ways: For instance, we could increase $\mu = (R+1)/LC$ by increasing R or decreasing LC. In the limit $LC \ll 1$, we have $\mu \gg 1$; then (15) reduces to $I_{\text{thresh}} \approx (LC)^{-1/2} \gg 1$. In this case it is easy to satisfy the stability condition.

In summary, the splay state is neutrally stable if $1 < I_b < I_{\text{thresh}}$; unstable if $I_b > I_{\text{thresh}}$; and does not exist if $I_b < 1$. Neutral stability is promoted by large values of R or small values of LC.

As a final caveat, we reiterate that all of the preceding analysis assumes that the load actually has a capacitance ($C \neq \infty$). Otherwise in the stationary state there is a constant *current* passing through the load, rather than a constant *charge* on the capacitor. This case requires a separate treatment, because the stationary density $\rho_0(\phi)$ is qualitatively different from that studied here. No additional techniques are required; we leave the details for the interested reader. The conclusions are that for $I_b > 1$ the splay state is always unstable for $L \neq 0$ and is always neutrally stable for $L = 0$. This last case, in which the load is purely resistive, has been considered before—see Tsang *et al.* [5] and Swift, Strogatz, and Wiesenfeld [10].

V. DISCUSSION

Our analysis accounts for the puzzling results of Nichols and Wiesenfeld [1]. The enormous degree of neutral stability is due to the decoupling of the higher harmonics in Eq. (9); that decoupling is itself a manifestation of the pure sinusoidal form of the Josephson current relation. Our theory also gives accurate predictions of the nontrivial Floquet multipliers, and explains why there are precisely four of them.

Related phenomena, including extensive neutral stability, have been seen previously in studies of various model systems with permutation symmetry and sinusoidal nonlinearities [22–24]. In particular, Golomb *et al.* [23] showed numerically that the neutral stability could be eliminated by including higher harmonics in the nonlinearities.

On the other hand, there must be more to this story—permutation symmetry and sinusoidal nonlinearity alone do not imply the neutral stability of splay states. As a counterexample, consider Josephson arrays where the individual junctions have *nonzero* capacitance, i.e., the McCumber parameter $\beta > 0$ [21], and the load is purely capacitive. For this case, Nichols and Wiesenfeld have found numerically that the splay state becomes linearly *stable,* with exponential contraction in all directions transverse to the orbit. This stabilization cannot be ascribed to a change in the symmetry or the nonlinearity; some other effects are at work here, and await understanding.

The analysis of arrays with $\beta > 0$ is going to require additional methods. The state of a junction is now given by *two* numbers, a phase ϕ and an angular velocity $\dot{\phi} = \omega$. In the infinite-N limit, the appropriate density is $\rho(\phi, \omega, t)$, which represents a density on a *cylinder.* In the stationary state, all the junctions execute identical limit-cycle oscillations on this cylinder, and are uniformly staggered in time around the limit cycle. Hence $\rho(\phi, \omega, t)$ collapses to a *singular* density supported on this limit cycle. The linearized system is therefore more subtle to analyze, and involves "generalized functions." Another difficulty is that the basic limit cycle cannot be written down explicitly. Work on this case is in progress.

There are also several unanswered questions about the analytical approach introduced in this paper. The underlying philosophy is that it is always easier to find fixed points and their eigenvalues than it is to find periodic orbits and their Floquet exponents. For the Josephson array studied here, the Floquet exponents of a finite-N system turned out to be closely approximated by the eigenvalues of an infinite-N system. Because of this relationship, we were in the unusual position of being able to obtain analytical information about all the Floquet multipliers. But we have offered no proof that the method works (although it clearly does). Can the method be proven to work in more general circumstances? And can one apply it to other globally coupled systems, such as laser arrays, electronic oscillator circuits, or multimode lasers?

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