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Classification of attractors for systems of identical coupled Kuramoto oscillators

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We present a complete classification of attractors for networks of coupled identical Kuramoto oscillators. In such networks, each oscillator is driven by the same first-order trigonometric function, with coefficients given by symmetric functions of the entire oscillator ensemble. For \( N \neq 3 \) oscillators, there are four possible types of attractors: completely synchronized fixed points or limit cycles, and fixed points or limit cycles where all but one of the oscillators are synchronized. The case \( N = 3 \) is exceptional; systems of three identical Kuramoto oscillators can also possess attracting fixed points or limit cycles with all three oscillators out of sync, as well as chaotic attractors. Our results rely heavily on the invariance of the flow for such systems under the action of the three-dimensional group of Möbius transformations, which preserve the unit disc, and the analysis of the possible limiting configurations for this group action. © 2014 AIP Publishing LLC.

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We consider identical Kuramoto oscillator networks in which each oscillator is represented by an angular variable, and the driving equation for each oscillator is the same first-order trigonometric function in this angular variable, with coefficients given by symmetric functions over the entire oscillator ensemble. Such systems can be used to model coupled pendula and certain types of Josephson junction arrays. We give a complete classification of the possible types of attractors for such systems, which is equivalent to classifying all possible long-term stable dynamical behavior. For \( N \neq 3 \) oscillators, we prove that there are four possible types of attractors: fixed points or limit cycles which have all of the oscillators in complete synchrony, or fixed points or limit cycles which have exactly one outlier; all but one of the oscillators are in sync. The case \( N = 3 \) is exceptional; systems of three identical Kuramoto oscillators can also possess attracting fixed points or limit cycles with all the oscillators out of sync, or can even have chaotic attractors. The essential ingredient in the derivation of our results is the invariance of the flow for such systems under the action of the three-dimensional group of Möbius transformations, which preserve the unit disc, together with an analysis of the possible limiting configurations of sets of points on the unit circle under this group action.

\section*{I. INTRODUCTION}

In the study of coupled oscillator networks, one often encounters systems where the individual oscillators are governed by an equation of the form

\[ \dot{\theta}_j = A + B \sin \theta_j + C \cos \theta_j, \]

where the coefficients \( A, B, C \) are usually obtained by averaging certain quantities over all or part of the ensemble of oscillators in the network, or may also depend non-autonomously on time. The classic example is the famous Kuramoto coupled oscillator model,\textsuperscript{4} the equations for a resistively coupled series array of Josephson junctions\textsuperscript{2} or a system of identical coupled nonlinear pendula also have this form.\textsuperscript{3} We call oscillators governed by equations of this type Kuramoto oscillators. In this paper, we focus on networks of identical Kuramoto oscillators and completely classify the possible long-term stable dynamics of such networks. More precisely, we consider an autonomous system of \( N \) oscillators governed by the equations

\[ \dot{\theta}_j = A + B \sin \theta_j + C \cos \theta_j, \quad j = 1, \ldots, N \]

where \( \theta_j \) is an angular variable (i.e., an element of \( \mathbb{R} / \mathbb{Z} \)) and the coefficients \( A, B, C \) are smooth, symmetric functions of \( \theta_1, \ldots, \theta_N \). In other words, the coefficients may be thought of as functions of the “unmarked” ensemble of points \( \{ e^{i \theta_1}, \ldots, e^{i \theta_N} \} \) on the unit circle \( S^1 \). The state space for this system is the \( N \)-fold torus \( T^N = (S^1)^N \), and the diagonal subspace \( \Delta \subset T^N \) defined by \( \theta_1 = \cdots = \theta_N \) mod \( 2\pi \) is clearly invariant under the flow for (1).

So any system of identical Kuramoto oscillators will have fully synchronized solutions, which either converge to fixed points or are periodic cycles. Perhaps not surprisingly, both of these types of fully synchronized states can be attracting (we will give examples later in this paper).

At the other extreme, systems of identical Kuramoto oscillators can also have “splay states,” which are periodic orbits in which each oscillator has the same evolution in time, but the oscillators are equally staggered in phase. In other words,

\[ \theta_j(t) = \phi \left( t - \frac{j}{N} \right), \quad j = 1, \ldots, N, \]

for some function \( \phi \) with period \( T > 0 \). The existence of splay states has been established under a variety of conditions. For example, if the functions \( A, B, C \) have the property that
\[ A + B \sin \theta_j + C \cos \theta_j > 0 \]

for all \((\theta_1, \ldots, \theta_N)\), then (1) has splay states.\(^4\)\(^5\) The stability of splay states is another matter; in fact, splay states are never asymptotically stable, as long as \(N \geq 4\); this is a consequence of our main theorem and has also been proved in a number of special cases.\(^6\)\(^–\)\(^11\) In fact, as we shall see, the asymptotically stable states for (1) are always completely synchronized, except for perhaps one exceptional oscillator, provided \(N \neq 3\).

The study of the stability of splay states for systems of the form (1) was the principal motivation for the present work. It has been observed since the early 1990’s that splay states for systems of the form (1) exhibit a remarkable degree of neutral stability. The first conceptual explanation of this phenomenon was given by Watanabe and Strogatz, who in an algebraic tour-de-force\(^b\) constructed \(N - 3\) constants of motion for a series array of \(N\) Josephson junctions, which is a special case of (1). The existence of these conserved quantities implies that the splay states must have at least \(N - 2\) neutral Floquet multipliers (including the neutral Floquet multiplier from the direction of the splay orbit itself). Long after their paper appeared, a much simpler explanation and construction of these conserved quantities was given in terms of the action on the torus \(T^N\) of the 3-dimensional group \(G\) of Möbius transformations, which preserve the unit disc in the complex plane. In general, a Möbius transformation is a fractional linear map of the form

\[ M(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0, \]

which extends to a holomorphic diffeomorphism on the extended complex plane \(\mathbb{C}\). We are interested in the Möbius transformations, which map the unit circle to itself preserving orientation; these have the form

\[ M(z) = \frac{z + \alpha}{1 + \overline{\alpha}z}, \quad |\alpha| < 1, \quad |\beta| = 1. \]

Such Möbius maps form a three-dimensional group \(G\) (not coincidentally, the dimension of \(G\) is the exceptional value of \(N\) in our theorem). The group \(G\) acts on the \(N\)-fold torus \(T^N\) in a natural way; if we denote a point \(p \in T^N\) by a vector \((\xi_1, \ldots, \xi_N)\), with \(|\xi_j| = 1\), and \(M \in G\), then we define the action by

\[ Mp = (M(\xi_1), \ldots, M(\xi_N)). \]

Thus, for each point \(p \in T^N\), its \(G\)-orbit \(Gp\) under this action is given by

\[ Gp = \{Mp | M \in G\}. \]

Let \(\Phi_t\) denote the time-\(t\) flow for (1); then \(\Phi_t(p) \in Gp\) for all \(t \in \mathbb{R}\); in other words, the trajectories of the system (1) are constrained to lie within the \(G\)-orbits for the Möbius group action. This can be seen in a number of different ways; for example, see Ref. 10. For the sake of completeness, we outline the proof of this important assertion.

Let \(p \in T^N\), and let \(a(t), b(t),\) and \(c(t)\) denote the values of the coefficient functions \(A, B,\) and \(C\) at the point \(\Phi_t(p)\). Using the relation \(\zeta_j = e^{i\theta_j}\), which implies \(\dot{\zeta}_j = i\zeta_j \dot{\theta}_j\), we see that each coordinate \(\zeta_j\) evolves according to the equation

\[ \dot{\zeta}_j = i\zeta_j \left( a(t) + \frac{b(t)}{2i} \left( \frac{\zeta_j}{\zeta_j^2} + \frac{c(t)}{2} \left( \frac{1}{\zeta_j} + \frac{1}{\zeta_j^2} \right) \right) \right) = \frac{1}{2} \left( b(t) + ic(t) \right) \zeta_j^2 + i a(t) \zeta_j + \frac{1}{2} \left( -b(t) + ic(t) \right). \]
This differential equation on $S^1$ extends to a Riccati equation on $\mathbb{C}$, which has the special form
\[
\dot{z} = Z(t)z^2 + ia(t)z - \overline{Z(t)},
\]
where $Z(t) = \frac{1}{2}(b(t) + ic(t))$. If we make the change of variables $w = \frac{1}{z}$, then this equation transforms to
\[
\dot{w} = \overline{Z(t)}w^2 - ia(t)w - Z(t).
\]
This shows that the equation extends to a holomorphic differential equation on the extended complex plane $\hat{\mathbb{C}}$. The time-$t$ flow $M_t$ for this equation must be a holomorphic diffeomorphism of $\hat{\mathbb{C}}$; i.e., a Möbius transformation.\(^\text{12}\)

Furthermore, this Riccati equation preserves the unit circle, so $M_t$ must be an orientation-preserving self-map of the unit circle (to check this, show $(\frac{2}{1}) = 0$ if $z \overline{z} = 1$). Hence, we see that
\[
\Phi_t(p) = (M_t(z_1), \ldots, M_t(z_N)) = M_tp \in Gp,
\]
which proves the claim.

So we see that the state space $T^N$ is decomposed into orbits under the action of $G$, which are all invariant under the flow for (1). For $N \neq 3$, this might seem to rule out the possibility of any attractors whatsoever; if two points in $T^N$ are in different $G$-orbits, no matter how close the points are to each other, then their respective trajectories will forever remain on their distinct, non-intersecting orbits. However, the $G$-orbits are not compact, so it is indeed possible for the trajectories of two points on distinct $G$-orbits to have the same limit sets. In fact, understanding the possible limiting values of trajectories, or equivalently, the orbital boundaries defined by $(Gp)^* = \mathcal{G}p - Gp$, is the key to proving our main result. (Note that $(Gp)^*$ is not the same set as the topological boundary of $Gp$, unless $Gp$ is an open set, which is usually not the case.)

The organization of this paper is as follows. We present the proof of our theorem in Sec. II, and then present examples and constructions for the various possible types of attractors when $N \neq 3$. We follow with examples and constructions for the exceptional behavior for the case $N = 3$ and some concluding remarks.

II. PROOF OF THEOREM

The proof of our theorem is based on an analysis of the possible limiting configurations of the Möbius group orbits. As observed above, at first glance, it might seem that when $N > 3$, (1) should not have any attracting states at all, since the state space is foliated into a family of three-dimensional invariant manifolds, and so all fixed points or periodic orbits would have at least $N - 3$ neutral directions. This reasoning is in fact correct for states in which at least three of the oscillator phases are distinct. But the Möbius group orbits are not compact subsets of the torus $T^N$, so it is indeed possible for the flows of points on different $G$-orbits to converge to the same fixed point or limit cycle. In lemmas 1 and 2, we show that for a point $p \in T^N$ with all distinct coordinates, the limiting positions of its orbit $Gp$ consist exactly of the set $\Lambda$. This implies that all attractors must lie in $\Lambda$, assuming $N > 3$. (In fact, our attractor classification holds true fairly trivially for $N = 1$ or 2, so it is only the case $N = 3$ that is exceptional.) This essentially completes the proof of the theorem; all that remains is to prove that attractors in $\Lambda$ must be fixed points or limit cycles.

We begin with the lemmas that characterize the orbital boundaries. Note that all limits are understood in terms of the standard metric on the Riemann sphere $\hat{\mathbb{C}}$.

**Lemma 1.** Suppose $(M_n)_{i}$ is a sequence of Möbius transformations, $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ are distinct points, and $\lim_{n \to \infty} M_n(z_j) = w_j \in \hat{\mathbb{C}}$, where the $w_j$ are distinct. Then, there exists an unique Möbius map $M$ such that
\[
\lim_{n \to \infty} M_n(z) = M(z) \quad \text{for all} \quad z \in \hat{\mathbb{C}}.
\]

**Proof.** We use some basic properties of Möbius transformations (see Ref. 12). There is a unique Möbius map $M$ which maps $z_j$ to $w_j$; if $z \neq z_j$, then $M(z)$ is uniquely determined by the cross-ratio equation
\[
\frac{M(z) - w_2}{M(z) - w_3} = \frac{z - z_2}{z - z_3} = \frac{z_1 - z_2}{z_1 - z_3}.
\]

We wish to prove that $\lim_{n \to \infty} M_n(z) = M(z)$. If not, then $(M_n(z))$ must have a subsequence with limit $\neq M(z)$, since $\hat{\mathbb{C}}$ is compact. Hence, we can assume without loss of generality that $w = \lim_{n \to \infty} M_n(z)$ exists. Möbius maps preserve cross-ratios, so we have the relation
\[
\frac{M_n(z) - M_n(z_2)}{M_n(z) - M_n(z_3)} = \frac{M_n(z_1) - M_n(z_2)}{M_n(z_1) - M_n(z_3)} = \frac{z - z_2}{z - z_3} = \frac{z_1 - z_2}{z_1 - z_3}.
\]

Taking limits in this equation gives
\[
\frac{w - w_2}{w - w_3} = \frac{z - z_2}{z - z_3} = \frac{z_1 - z_2}{z_1 - z_3},
\]
and therefore $w = M(z)$.

This lemma tells us a lot about the orbital boundaries $(Gp)^*$. For example, suppose $p \in T^N$ has at least three distinct coordinates, and $q \in \mathcal{G}p$. Then, $q = \lim_{n \to \infty} M_n p$ for some sequence $M_n \in G$; if $q$ has at least three distinct coordinates, Lemma 1 implies $q = \mathcal{G}p$ for some $M \in G$, so in fact $q \in \mathcal{G}p$. Hence, any $q \in (Gp)^*$ can have at most two distinct coordinates. The next lemma shows that we can say even more.

**Lemma 2.** Suppose $(M_n)_{i}$ is a sequence of Möbius transformations, $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ are distinct points, and $\lim_{n \to \infty} M_n(z_j) = w_j \in \hat{\mathbb{C}}$, where $w_1, w_2 \neq w_3$. Then,
\[
\lim_{n \to \infty} M_n(z) = w_1 \quad \text{for all} \quad z \in \hat{\mathbb{C}} \quad \text{such that} \quad z \neq z_j.
\]

**Proof.** Suppose $z \neq z_j$, then the right side of the cross-ratio Eq. (2) is not 0 or $\infty$, and does not depend on $n$. On the left side, we have
\[
\lim_{n \to \infty} \frac{M_n(z_1) - M_n(z_2)}{M_n(z_1) - M_n(z_2)} = 0.
\]

Therefore, we must also have
\[
\lim_{n \to \infty} \frac{M_n(z) - M_n(z_2)}{M_n(z) - M_n(z_3)} = 0,
\]

which implies that
\[
\lim_{n \to \infty} M_n(z) = \lim_{n \to \infty} M_n(z_2) = w_1.
\]

Consequently, we see that if \( p \in T^N \) has all distinct coordinates and \( N \geq 3 \), then any \( q \in (Gp)^* \) must have identical coordinates, with at most one exception; in other words, \((Gp)^* \subset \bar{\Delta} \). It is also not hard to see that equality holds in this relation, since given any \( z_1 \) and \( w_1 \neq w_2 \in \tilde{\mathbb{C}} \), there exist Möbius maps \( M_z \) such that \( M_z(z_1) = w_1 \) and \( \lim_{n \to \infty} M_n(z) = w_2 \) for all \( z \neq z_1 \). (To see this, it is sufficient to consider the special case \( z_1 = 0, w_1 = 0, \) and \( w_2 = \infty \). Then, the maps \( M_z(z) = nz \) have the desired property.) Hence, the orbital boundary \((Gp)^* = \bar{\Delta} \) if \( p \) has all distinct coordinates. Now, we can prove the main theorem:

**Theorem.** The attractors for (1) are fixed points or limit cycles in \( \bar{\Delta} \), provided \( N \neq 3 \).

**Proof.** Suppose \( N \geq 4 \) and \( A \) is an attractor for (1), then there is a subset \( U \subset T^N \) with positive measure such that \( A = \omega(p) \) for all \( p \in U \). Since \( U \) has positive measure, we can choose \( p, p' \in U \) such that both points have all distinct coordinates and \( Gp \cap Gp' = \emptyset \). Then \( A = \omega(p) \subset Gp \) and also \( A = \omega(p') \subset Gp' \), so \( A \subset \bar{\Delta} \).

To complete the proof, it suffices to show that any compact invariant subset \( A \subset \bar{\Delta} \) which has a dense forward orbit must be a fixed point or periodic orbit (this will also take care of the case \( N = 2 \) when \( N \leq 2 \). So assume the forward orbit of \( p \in A \) is dense in \( A \) and \( p \) is not a fixed point. Suppose \( p \) is in the diagonal \( \Delta \), which is invariant under (1) and diffeomorphic to \( S^1 \). Then, \( p \) must be periodic; otherwise, \( \lim_{n \to \infty} \Phi_t(p) = p' \) exists and is a fixed point, so \( A \) is the closed interval on the circle from \( p \) to \( p' \) in the direction of the flow, which is not invariant under \( \Phi_t \) for \( t < 0 \).

If \( p \notin \Delta \), then without loss of generality we can express \( p \) in the form \( p = (\zeta_1, \ldots, \zeta_1, \zeta_2) \) with \( \zeta_1 \neq \zeta_2 \). The set
\[
X_0 = \{ (\zeta_1, \ldots, \zeta_1, \zeta_2) \mid \zeta_1 \neq \zeta_2 \} \subset T^N
\]
is also invariant under (1) and is diffeomorphic to the open annulus \( S^1 \times (0,1) \), which is a planar domain. Consider any arc \( \gamma \) in \( X_0 \) containing \( p \) and transverse to the orbit \( \Phi_t(p) \) at \( p \). Since \( \Phi_t(p) \in A \) for \( t < 0 \), the orbit must return to \( \gamma \) in forward time, so there exists a smallest \( T > 0 \) such that \( \Phi^T(p) = \Phi_T(p) \in \gamma \) (see Figure 1). We claim that \( p' = p \), which will finish the proof. If not, then the simple closed curve \( C \) consisting of the closed arc along the orbit of \( p \) from \( t = 0 \) to \( t = T \) and the segment of \( \gamma \) joining \( p \) and \( p' \) is a Jordan curve, which divides \( X_0 \) into two disconnected regions \( V, W \); one of these, say \( V \), contains all points \( \Phi_t(p) \) for \( t > T \) and the other, say \( W \), contains all points \( \Phi_t(p) \) for \( t < 0 \). But then the closure of the forward orbit of \( p \) is contained in \( V \cup C \), which is disjoint from \( W \), and hence does not contain \( \Phi_t(p) \) for \( t < 0 \), which contradicts the backward-time invariance of \( A \). \[\square\]

### III. EXAMPLES AND CONSTRUCTIONS

We now present examples which illustrate that the four types of attractors that are possible according to our theorem do in fact occur.

#### A. Synchronized fixed points

These are easy to exhibit for any value of \( N \); the uncoupled model given by
\[
\dot{\theta}_j = -\sin \theta_j \quad j = 1, \ldots, N
\]
has an attracting fixed point \( p \) given by \( \theta_j = 0 \) for all \( j \).

#### B. \((N-1,1)\) fixed points

Consider the special case of (1) defined by
\[
\dot{\theta}_j = \cos \theta_j + \sum_{k=1}^{N-1} f(\theta_k),
\]
where \( f \) is some \( 2\pi \)-periodic function to be determined later. We will choose \( f \) so that the point \( \theta_0 = \cdots = \theta_{N-1} = \pi/4 \), \( \theta_N = -\pi/4 \) is an attracting fixed point for this system. For \( p \) to be a fixed point, \( f \) must satisfy the equation
\[
(N-1)f(\frac{\pi}{4}) + f(\frac{-\pi}{4}) + \sqrt{2} = 0,
\]
which can easily be achieved if necessary by adjusting \( f \) by a constant.

The linearized system at \( p \) is given by
\[
\dot{x}_j = -\frac{\sqrt{2}}{2} x_j + f'(\frac{\pi}{4}) \sum_{k=1}^{N-1} x_k + f'(\frac{-\pi}{4}) x_N, \quad j = 1, \ldots, N-1
\]
\[
\dot{x}_N = \frac{\sqrt{2}}{2} x_N + f'(\frac{\pi}{4}) \sum_{k=1}^{N-1} x_k + f'(\frac{-\pi}{4}) x_N.
\]

This system has invariant subspaces given by

![FIG. 1. This figure illustrates the final step in the proof of the theorem. If \( p \neq p' \), then the forward orbit of \( p \) cannot be dense in the attractor \( A \), since \( A \) also contains the backward orbit of \( p \) which lies outside of \( V \).](image-url)
FIG. 2. Plot of $P_N$, the relative size of the basin of attraction of the $(N-1,1)$ fixed points for the system (3) with $f(\theta)$ given by (4), as a function of $N$.

$V_1 = \{ (x_1, \ldots, x_{N-1}, 0) | \sum x_k = 0 \}$ and $V_2 = \{ (x_1, \ldots, x, y) \}$, which are disjoint and together span $\mathbb{R}^N$. The system on $V_1$ uncouples and reduces to

$$\dot{x}_j = -\frac{\sqrt{2}}{2} x_j, \quad j = 1, \ldots, N-1,$$

which has $-\sqrt{2}/2$ as an $(N-2)$-fold eigenvalue. The system on $V_2$ is given by the $2 \times 2$ matrix

$$L = \begin{pmatrix} -\frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix} + \begin{pmatrix} (N-1)f'(\frac{\pi}{4}) & f'(\frac{-\pi}{4}) \\ (N-1)f'(\frac{-\pi}{4}) & f'(\frac{-\pi}{4}) \end{pmatrix}.$$

Suppose we require that $f'(\pi/4) = 0$; then $L$ is upper-triangular with eigenvalues

$$\lambda = -\frac{\sqrt{2}}{2}, \quad \frac{\sqrt{2}}{2} + f'(\frac{-\pi}{4}).$$

Therefore, $\rho$ is an attracting fixed point provided

$$f'(\frac{\pi}{4}) = 0, \quad f'(\frac{-\pi}{4}) < -\frac{\sqrt{2}}{2}.$$ 

So, for example, the function

$$f(\theta) = \frac{2N-3}{2N}\sqrt{2} - \cos \theta - \sin \theta \quad (4)$$

satisfies all the necessary requirements. So (1) can have type $(N-1,1)$ attracting fixed points for any $N$.

Numerical simulations of the system of $N$ oscillators described by (3) with $f(\theta)$ given by (4) reveal some unexpected properties of these fixed points. To measure the relative size of the basin of attraction of the $(N-1,1)$ fixed points, we determine the probability $P_N$ that $\Phi_t(p_0)$ flows to any of the $N$ equivalent fixed points of type $(N-1,1)$ starting from an arbitrary initial $p_0$ with $N$ randomly chosen $\theta_j$. We find that $P_3 \approx 0.875$, and that $P_N$ decreases rapidly as a function of $N$, although the rate of decrease is slower than exponential. Figure 2 shows the scaling of $P_N$ on a log-log plot, which appears to be consistent with a power-law decrease in the size of the basin of attraction for larger $N$.

Moreover, we can numerically explore the structure of the basin of attraction of the $(N-1,1)$ fixed points in $\mathbb{T}^N$ by plotting two-dimensional sections of the basin of attraction inside a torus $\mathbb{T}^2$. To do this, we choose the initial phases of the first two oscillators $(\theta_1, \theta_2)$ randomly in $\mathbb{T}^2$, and then set the initial phases of the $N-2$ remaining oscillators to be $\theta_j = c_0 + c_1\theta_1 + c_2\theta_2$, where $c_0$ is a fixed random phase and $c_1$ and $c_2$ are fixed random integers in the range $[-5,5]$. The shaded areas in Figures 3(a) and 3(b) display these sections for $N = 5$ and $N = 7$, respectively. The different shades correspond to different fixed points, depending on which of the $N$ oscillators flows to the exceptional value $-\pi/4$. Figure 3(c) shows an expanded region from Figure 3(b). While each of

FIG. 3. This figure shows two-dimensional sections of the basins of attraction of the $(N-1,1)$ fixed points for the system (3) with $f(\theta)$ given by (4), with $N = 5$ (A) and $N = 7$ (B) and (C). Different shades indicate which of the $N$ oscillators flows to the exceptional value $-\pi/4$. Panel (C) is a blow up of a small region in (B).
the $N$ basins of attraction in $T^N$ must be contractible, these sections in $T^2$ seem to have remarkably complicated structure.

C. Synchronized limit cycles

Suppose $\Theta(t) = (\theta_1(t), ..., \theta_N(t))$ is a solution to (1) then the linearization along this trajectory is given by the system

$$
\dot{x}_j = (B(\Theta(t)) \cos \theta(t) - C(\Theta(t)) \sin \theta(t)) x_j.
$$

This subspace accounts for the $N - 1$ Floquet exponents for the trajectory $\Theta(t)$. The remaining zero Floquet exponent comes from the flow along the trajectory itself; i.e., the flow on the diagonal $\Delta$.

To construct an example of an attracting limit cycle, we can take

$$
A = 1, \quad B = 0, \quad C = \frac{1}{N} \sum_{k=0}^{N-1} \sin \theta_k.
$$

Then, the flow on the diagonal subspace is given by

$$
\dot{\theta} = 1 + \sin \theta \cos \theta,
$$

so $\dot{\theta} \geq 1/2$ and hence the system has a synchronized periodic orbit $\Theta(t) = (\theta(t), ..., \theta(t))$ with period $T > 0$. The linearization on the subspace $V$ reduces to

$$
\dot{x}_j = -(\sin^2 \theta(t)) x_j,
$$

so the $N - 1$ Floquet exponents are each equal to

$$
\mu = -\int_0^T \sin^2 \theta(t) \, dt < 0.
$$

Therefore, this synchronized periodic orbit is an attracting limit cycle.

D. $(N-1,1)$ limit cycles

In this case, we will begin with the $(N-1,1)$ periodic solution

$$
\Theta(t) = \left(t, ..., t, t - \frac{\pi}{2}\right)
$$

and construct the system so that $\Theta(t)$ is an attracting limit cycle. We set the coefficient function $A=0$. Let $b(t), c(t)$ denote the values of the coefficient functions $B, C$ respectively at $\Theta(t)$. Then, the functions $b(t), c(t)$ must satisfy the equations

$$
b(t) \sin t + c(t) \cos t = 1 \quad \text{and} \quad b(t) \sin \left(t - \frac{\pi}{2}\right) + c(t) \cos \left(t - \frac{\pi}{2}\right) = 1,
$$

or equivalently

$$
\begin{pmatrix}
\sin t & \cos t \\
-\cos t & \sin t
\end{pmatrix}
\begin{pmatrix} b(t) \\ c(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
$$

Inverting this system gives

$$
b(t) = \sin t - \cos t, \quad c(t) = \sin t + \cos t.
$$

The functions $B$ and $C$ will be constructed to be symmetric in the $\theta_j$, so each will have the same first partial derivatives with respect to the first $N-1$ variables along this trajectory. Therefore, the $(N-2)$-dimensional invariant subspace defined by

$$
V_1 = \left\{ (x_1, ..., x_{N-1}, 0) \mid \sum_{k=1}^{N-1} x_k = 0 \right\}
$$

is invariant for (5); and on this subspace, each coordinate $x_j$ is governed by the simpler equation

$$
\dot{x}_j = (b(t) \cos \theta_j(t) - c(t) \sin \theta_j(t)) x_j
= ((\sin t - \cos t) \cos t - (\cos t + \sin t) \sin t) x_j = -x_j.
$$

Hence, we will have $N-2$ negative Floquet exponents $\mu = -2\pi$.

It remains to construct the symmetric functions $B, C$ on $T^N$, so that $\Theta(t)$ is an attracting limit cycle. We begin on the subspace of $\Delta$ defined by

$$
T^2 = \{ (\theta_1, ..., \theta_1, \theta_2) \},
$$

which is a two-dimensional torus containing the periodic cycle $\Theta(t)$. Let

$$
\begin{align*}
\dot{\theta}_1 &= f_1(\theta_1, \theta_2) \\
\dot{\theta}_2 &= f_2(\theta_1, \theta_2)
\end{align*}
$$

be any smooth system on $T^2$ for which $\Theta(t)$ is an attracting limit cycle; clearly such systems exist. We can require that $f_1$ and $f_2$ are supported inside a thin tubular neighborhood of the trajectory $\Theta(t)$ in $T^2$, say of the form

$$
U = \{ (\theta_1, ..., \theta_1, \theta_2) \mid |\theta_1 - \theta_2 - \pi/2| < \epsilon \},
$$

where $0 < \epsilon < \pi/2$.

Next, we can express this system on $T^2$ in the form

$$
\begin{align*}
\dot{\theta}_1 &= B \sin \theta_1 + C \cos \theta_1 \\
\dot{\theta}_2 &= B \sin \theta_2 + C \cos \theta_2
\end{align*}
$$

by solving the system
for \( B \) and \( C \). Note that the determinant of this system is 
\( \sin(\theta_1 - \theta_2) \), which is \( \approx 1 \) on \( U \), so there is no problem inverting this system. This defines the coefficient functions \( B \) and \( C \) on the subspace \( T^2 \).

Finally, we must extend the functions \( B \) and \( C \) from \( T^2 \) to symmetric functions on all of \( T^N \). Extend \( B \) and \( C \) to smooth functions \( F \) and \( G \) respectively on \( T^N \) with support in the neighborhood of \( \Theta(t) \) in \( T^N \) given by
\[
\tilde{U} = \left\{ (\theta_1, \ldots, \theta_{N-1}, \theta_N) \mid |\theta_j - \theta_k| < \epsilon, |\theta_j - \theta_N - \pi/2| < \epsilon, 1 \leq j, k \leq N - 1 \right\}.
\]

Next, symmetrize \( F \) and \( G \) by averaging over all \( N! \) permutations of the coordinates to produce symmetric functions \( \tilde{F}, \tilde{G} \) on \( T^N \). If \( p = (\theta_1, \ldots, \theta_1, \theta_2) \in U \subset T^2 \), then \( F \) and \( G \) will vanish on all permutations of the coordinates that move the last coordinate, since these permutations move \( p \) out of the neighborhood \( \tilde{U} \). The remaining \((N - 1)! \) permutations fix \( p \), so
\[
\tilde{F}(p) = \frac{1}{N} B(p), \quad \tilde{G}(p) = \frac{1}{N} C(p).
\]

Hence, the symmetric functions \( NF, NG \) are extensions of \( B, C \) from \( T^2 \) to \( T^N \), which completes the construction.

The periodic cycle \( \Theta(t) \) is an attracting limit cycle for the system restricted to the two-dimensional torus \( T^2 \), and so the linearization has a negative Floquet exponent on the invariant subspace
\[
V_2 = \{ (x, \ldots, x, y) \},
\]
which is disjoint from \( V_1 \). Hence, all the (nontrivial) \( N - 1 \) Floquet exponents for \( \Theta(t) \) are negative, and therefore \( \Theta(t) \) is an attracting limit cycle.

IV. THE CASE \( N = 3 \)

Consider any smooth system on the torus \( T^3 \) given by
\[
\dot{\theta}_j = F_j(\theta_1, \theta_2, \theta_3), \quad j = 1, 2, 3.
\]

We say this system is symmetric or \( S_3 \)-invariant if it has the property that for any trajectory of the system, all its coordinate permutations are also trajectories. It is easy to see that this system is \( S_3 \)-invariant if and only if

(i) The function \( f = F_1 \) satisfies \( f(\theta_1, \theta_2, \theta_3) = f(\theta_1, \theta_3, \theta_2) \);

(ii) We have \( F_2(\theta_1, \theta_2, \theta_3) = f(\theta_2, \theta_3, \theta_1) \) and \( F_3(\theta_1, \theta_2, \theta_3) = f(\theta_3, \theta_1, \theta_2) \).

So an \( S_3 \)-invariant system is completely determined by a single function \( f \) on \( T^3 \), which must be invariant under transposition of the last two coordinates.

Symmetric systems on \( T^3 \) can exhibit the same kinds of dynamical behavior as general three-dimensional systems. For example, the system defined by
\[
\dot{\theta}_j = -\sin 3\theta_j, \quad j = 1, 2, 3
\]
has an attracting fixed point \((0, 2\pi/3, 4\pi/3)\) with all coordinates distinct. We can also construct symmetric systems with attracting splay states, as follows. Suppose the symmetric system \( \dot{\theta}_j = F_j(\theta_1, \theta_2, \theta_3) \) on \( T^3 \) has an attracting synchronized limit cycle \((\theta(t), \theta(t), \theta(t))\) with period \( T > 0 \). Then, the symmetric system given by
\[
\dot{\theta}_j = \frac{1}{3} F_j(3\theta_1, 3\theta_2, 3\theta_3)
\]
has an attracting splay limit cycle with period \( 3T \), given by
\[
\left( \frac{1}{3} \theta(t), \frac{1}{3} \theta(t) + \frac{2\pi}{3}, \frac{1}{3} \theta(t) + \frac{4\pi}{3} \right).
\]

We can even construct an \( S^3 \)-invariant system with a chaotic attractor. Let \( U \) be an open set in \( T^3 \) that is diffeomorphic to an open ball in \( \mathbb{R}^3 \), with the property that \( \sigma U \cap U = \emptyset \) for any permutation \( \sigma \in S_3 \) except the identity. Consider any dynamical system on \( U \) with the property that the defining functions have compact support in \( U \). Then, this system can be extended to all of \( T^3 \) by summing the functions over all permutations of the coordinates, and this does not change the functions on \( U \). So any dynamical behavior that is exhibited in a bounded region in \( \mathbb{R}^3 \) can be realized in an \( S_3 \)-invariant system.

As we shall soon see, any \( S_3 \)-invariant system can be expressed in the form of a coupled identical Kuramoto oscillator system, so \( S_3 \)-invariant systems and three-dimensional identical Kuramoto oscillator systems are one and the same. Hence, all the dynamic phenomena illustrated above can occur for these systems when \( N = 3 \). We conclude this section with the proof of this assertion.

**Proposition.** Any \( S_3 \)-invariant system on \( T^3 \) can be expressed in the form (1).

**Proof.** Let \( f(\theta_1, \theta_2, \theta_3) \) define an \( S_3 \)-invariant system. We need to construct symmetric functions \( A, B, C \) on \( T^3 \) that satisfy the system
\[
\begin{pmatrix}
1 & \sin \theta_1 & \cos \theta_1 \\
1 & \sin \theta_2 & \cos \theta_2 \\
1 & \sin \theta_3 & \cos \theta_3
\end{pmatrix}
\begin{pmatrix}
A \\
B \\
C
\end{pmatrix}
= \begin{pmatrix}
f(\theta_1, \theta_2, \theta_3) \\
f(\theta_2, \theta_3, \theta_1) \\
f(\theta_3, \theta_1, \theta_2)
\end{pmatrix}.
\]

The \( 3 \times 3 \) matrix \( L \) above has
\[
\det L = 4 \sin \left( \frac{\theta_1}{2} - \frac{\theta_2}{2} \right) \sin \left( \frac{\theta_2}{2} - \frac{\theta_3}{2} \right) \sin \left( \frac{\theta_3}{2} - \frac{\theta_1}{2} \right),
\]
which vanishes precisely on the augmented diagonal \( \tilde{\Delta} \). Note that \( \det L \) is invariant under even permutations of the \( \theta \) and changes by \(-1\) under transpositions. Away from \( \Delta \), we can solve for \( A, B, C \):
This determines the coefficient functions $A, B, C$ off of $\hat{A}$, and it is easy to see that $A, B$ and $C$ are symmetric in the $\theta_j$. For example,

$$A = \frac{\sin(\theta_2 - \theta_3) f(\theta_1, \theta_2, \theta_3) + \sin(\theta_3 - \theta_1) f(\theta_2, \theta_3, \theta_1) + \sin(\theta_1 - \theta_2) f(\theta_3, \theta_1, \theta_2)}{\det L},$$

and we see that transposing any two of the $\theta_j$ changes the sign of both the numerator and denominator, so $A$ is invariant under transpositions and hence all of $S_3$. The same clearly holds for $B$ and $C$. (The requirement that $f$ is invariant under transposition of the last two variables is crucial here.)

Finally, we must prove that the functions $A, B, C$ extend smoothly to the entire state space $T^3$. This is a consequence of the following lemma, together with the fact that $(\sin x)/x$ extends to a smooth function on $\mathbb{R}$.

**Lemma 3.** Let $f$ be a smooth function on $\mathbb{R}^3$. Then, the function $g$ defined by

$$g(x, y, z) = \frac{f(x, y, z)}{(x - y)(y - z)(z - x)}$$

extends to a smooth function on $\mathbb{R}^3$ if $f$ vanishes on the set \{(x − y)(y − z)(z − x) = 0\}.

We defer the proof of this technical lemma to the end of this section. To complete the proof of our proposition, it suffices to observe that the numerators in the expressions for the coefficient functions $A, B,$ and $C$ vanish whenever two of the $\theta_j$ are equal, which is clear from inspection.

To make this more concrete, we give two examples of calculations of the coefficients $A, B,$ and $C$, which explicitly show that these functions extend smoothly to the entire state space $T^3$, as guaranteed by Lemma 3. For the system (7), which has an attracting fixed point with three distinct angles, the function $f$ is given by $f(\theta_1, \theta_2, \theta_3) = -\sin 3\theta_1$. With the aid of Mathematica, we simplify the solution given by (9) and find that the coefficients are

$$A = 8 \sin(\theta_1 + \theta_2 + \theta_3) \cos\left(\frac{\theta_1 - \theta_2}{2}\right) \cos\left(\frac{\theta_1 - \theta_3}{2}\right) \cos\left(\frac{\theta_2 - \theta_3}{2}\right),$$

$$B = -\cos 2\theta_1 - \cos 2\theta_2 - \cos 2\theta_3 - \cos(\theta_1 + \theta_2) - \cos(\theta_2 + \theta_3) - \cos(\theta_3 + \theta_1)$$
$$+ \cos(2\theta_1 + \theta_2 + \theta_3) + \cos(\theta_2 + 2\theta_2 + \theta_3) + \cos(\theta_1 + \theta_2 + 2\theta_3),$$

$$C = -8(\sin \theta_1 + \sin \theta_2 + \sin \theta_3) \cos\left(\frac{\theta_1 + \theta_2}{2}\right) \cos\left(\frac{\theta_1 + \theta_3}{2}\right) \cos\left(\frac{\theta_2 + \theta_3}{2}\right).$$

The system with coefficients given by (6) with $N = 3$ has an attracting synchronized limit cycle. Therefore, following the construction given by (8), we see that the $S_3$-symmetric system with function $f$ given by

$$f(\theta_1, \theta_2, \theta_3) = \frac{1}{3} + \frac{1}{9} \sin 3\theta_1 + \sin 3\theta_2 + \sin 3\theta_3) \cos 3\theta_1$$

has an attracting splay limit cycle. Again using Mathematica, we simplify (9) and find that the coefficients are

$$A = \frac{1}{3} - \frac{8}{9} \cos(\theta_1 + \theta_2 + \theta_3)(\sin 3\theta_1 + \sin 3\theta_2 + \sin 3\theta_3) \cos\left(\frac{\theta_1 - \theta_2}{2}\right) \cos\left(\frac{\theta_1 - \theta_3}{2}\right) \cos\left(\frac{\theta_2 - \theta_3}{2}\right),$$

$$B = -\frac{8}{9} \cos \theta_1 + \cos \theta_2 + \cos \theta_3)(\sin 3\theta_1 + \sin 3\theta_2 + \sin 3\theta_3) \sin\left(\frac{\theta_1 + \theta_2}{2}\right) \sin\left(\frac{\theta_1 + \theta_3}{2}\right) \sin\left(\frac{\theta_2 + \theta_3}{2}\right),$$

$$C = \frac{1}{9} (\sin 3\theta_1 + \sin 3\theta_2 + \sin 3\theta_3) \cdot (\cos 2\theta_1 + \cos 2\theta_2 + \cos 2\theta_3 + \cos(\theta_1 + \theta_2) + \cos(\theta_1 + \theta_3) + \cos(\theta_2 + \theta_3)$$
$$+ \cos(2\theta_1 + \theta_2 + \theta_3) + \cos(\theta_1 + 2\theta_2 + \theta_3) + \cos(\theta_1 + \theta_2 + 2\theta_3)).$$

**Proof of Lemma 3.** The necessity of the condition is clear, since
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\[ f(x, y, z) = (x - y)(y - z)(z - x)g(x, y, z) \]

implies that \( f \) vanishes on \( \{(x - y)(y - z)(z - x) = 0\} \).

Conversely, if \( f \) vanishes on this set, then

\[
f(x, y, z) = f(x, x, z) + \int_{0}^{1} f_{2}(x, t, z) dt
= (y - x) \int_{0}^{1} f_{2}(x, x + s(y - x), z) ds,
\]

where \( f_{2} \) denotes the partial derivative with respect to \( y \). The function

\[ h(x, y, z) = - \int_{0}^{1} f_{2}(x, x + s(y - x), z) ds \]

is smooth on \( \mathbb{R}^{3} \), and we have \( f(x, y, z) = (x - y) h(x, y, z) \). Now \( h \) vanishes on the set \( \{(y - z)(z - x) = 0, x - y \neq 0\} \), and hence on its closure \( \{(y - z)(z - x) = 0\} \). Repeating the argument above, we see that \( h(x, y, z) \) extends to a smooth function \( k(x, y, z) \) on \( \mathbb{R}^{3} \), and similarly \( k(x, y, z) / (z-x) \) extends to a smooth function \( g(x, y, z) \) on \( \mathbb{R}^{3} \).

\[ \square \]

\section{V. CONCLUDING REMARKS}

As we have shown above the attractors for systems of identical Kuramoto oscillators have a simple classification; they must be either fixed points or limit cycles, and must be completely synchronized with the possible exception of one oscillator. This classification holds for systems with any number \( N \) of oscillators except \( N = 3 \); as we have seen, when \( N = 3 \) such systems can also possess strange attractors. As we mentioned in the introduction, this classification does not imply that the dynamics of systems of identical Kuramoto oscillators is always inherently simple. One source of complexity could be the basins of attraction of the attractors; as we saw above, it is possible for the attracting fixed points for these systems to have very complicated basins of attraction. Another source of complex behavior is that the trajectories for these systems need not converge to attractors at all, but instead remain inside the (generically) three-dimensional Möbius group orbits. This happens, for example, with the splay state trajectories for \( N \geq 3 \). The dynamics within these three-dimensional orbits can exhibit chaotic behavior; this can be shown by modifying the construction of chaotic attractors in the \( N = 3 \) case given in Sec. IV. In this case, the system can have an attractor for the dynamics restricted to the Möbius orbits, which will be continuously deformed as one continuously moves off this Möbius orbit to nearby Möbius orbits. This phenomenon was observed for the splay orbits in Ref. 6; the perfectly symmetric splay trajectories perturb to closed trajectories, which lie on different Möbius orbits from the splay trajectory. This would also be the case for strange attractors within the Möbius orbits; they too would form an \((N - 3)\)-dimensional continuous family of attractors, parameterized by the Möbius orbits. All of this underscores the observation that even though the attractors for systems of identical Kuramoto oscillators are extremely simple, these systems can exhibit remarkably subtle dynamic behavior.

Our analysis in this paper relies heavily on the assumption of identical oscillators, which does not hold for networks which have either some interesting topology, like chains or lattices of oscillators, or some other source of heterogeneity, like varying natural frequencies. But the methods we used, especially the strong constraints on dynamics due to the invariance under the Möbius group, do in fact apply to these more general systems. Indeed, we have already seen this for a family of Kuramoto oscillators with varying natural frequencies subject to non-autonomous periodic forcing.\(^{13}\) We proved that the long-term average frequency considered as a function of each oscillator’s natural frequency is never a “devil’s staircase”: it may have plateaux at integer multiples of the forcing frequency, but we prove it is strictly increasing between these plateaux. This is of course in sharp contrast to the case for more general periodically forced oscillators.

The simplest system on which to continue this line of inquiry might be a two-population model in which each Kuramoto oscillator belongs to one of two groups, and all the oscillators in each group are governed by the same equations. Such systems have been studied quite a bit in the literature, and can exhibit fascinating dynamics, including “chimera” states; see Refs. 14–17. In this context, there is a natural action of the direct product group \( G \times G \), where \( G \) is the usual Möbius group; the action is just the natural product action on the two populations. This will now constrain the dynamics to lie on the (generically) six-dimensional group orbits, so we would expect the group invariance to have a profound influence on the dynamic behavior, and perhaps lead to similar limitations on the possible attractors for such systems. And of course, these observations generalize naturally to more complicated multi-population models. So our hope is that the work in this paper is only part of the beginning of the exploration of group-theoretic methods in the study of coupled oscillator systems.


