Curvature, frame fields, and the Gauss-Bonnet theorem

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The question

We first examine the case of embedded surfaces. If \( \Sigma \subset \mathbb{R}^3 \) is a smooth, closed orientable surface embedded in three-space, then it has a well-defined outward unit normal vector field \( \hat{n} = (n_1, n_2, n_3) \) on it. Intuitively, the more curvaceous a surface is at a point, the more area a unit normal will trace out on \( S^2 \) as it moves about a small disk at that point. This intuition has a precise formulation as follows: regard such a unit normal vector field as a map \( G : \Sigma \to S^2 \), not surprisingly called the Gauss map. Then the pullback of the volume form to \( \Sigma \) is some function \( K \) times the volume form on \( \Sigma \), and \( K(p) \) is a measure of how curved \( \Sigma \) is at \( p \). \( K \) is called the Gaussian curvature.

**Theorem 1** (Gauss’s Theorema Egregium). \( K \) only depends upon the restriction of the ambient inner product to the tangent spaces of \( \Sigma \), and not upon the unit normal field. That is, the Gaussian curvature doesn’t depend upon the choice of embedding.

This theorem is the beginning of Riemannian geometry; the question now begs to be asked, “What if instead we had started off with only a notion of inner product on tangent spaces, and forgot about embeddings? What of the geometry would remain?” What remains is Riemannian geometry.

The genesis of the subject of my talk was another remarkable theorem of early Riemannian geometry. Let \( dA \) be the area form of \( \Sigma \), given by \( dA(V, W) = \hat{n} \cdot (V \times W) \). Then (cf. [13])

**Theorem 2** (Gauss-Bonnet). Let \( D \subset \Sigma \) be a simply-connected domain whose boundary \( \partial D \) consists of finitely many smooth curves. Then

\[
\int_{\partial D} k_g ds + \sum_j (\pi - \alpha_j) + \int_D KdA = 2\pi,
\]

where on each smooth curve \( \gamma \) (assumed to have unit speed) of \( \partial D \), \( k_g \) is the geodesic curvature (the component of \( \gamma' \) in the tangent plane), and the \( \alpha_j \) are the interior angles at which consecutive smooth boundary components meet.
Since this is slightly messy, we will focus instead upon von Dyck’s version, Theorem 3 (von Dyck 1888).

\[ \int_{\Sigma} KdA = 2\pi \chi(\Sigma). \]  

(2)

This manifests a significant interplay between geometry and topology. For instance, a surface of genus \( g \geq 2 \) has, in some sense, a majority of points with negative curvature. If the curvature is constant, then it must be negative. If this curvature is, for instance, \(-1\), then the area of \( \Sigma \) is an integer multiple of \( 4\pi \), and in fact is \( 4\pi (g(\Sigma) - 1) \). This marriage of geometry and topology is prevalent in low dimensions. However, some of the relationships still hold true in higher dimensions, and the Gauss-Bonnet theorem is one of these.

To state the general Gauss-Bonnet theorem, we must first define curvature on higher-dimensional Riemannian manifolds. Now, the Gaussian curvature generalizes easily for closed oriented hypersurfaces \( M^n \subset \mathbb{R}^{n+1} \) : we can still define an outward unit normal vector field on \( M \), which yields the Gauss map \( G : M \rightarrow S^n \). Pulling back the sphere’s volume form, we may define

\[ KdA = G^*d\sigma. \]  

(3)

Now, for any smooth map \( F : M^n \rightarrow N^n \) between oriented manifolds, if \( \omega \in \Omega^n(N) \) is a volume form, then

\[ \int_M F^*\omega = (\deg F) \int_N \omega, \]  

(4)

where \( \deg F \) is given by

\[ \begin{array}{c}
Z \xrightarrow{\mathbb{Z}^{n}} H^n(N;\mathbb{Z}) \xrightarrow{F^*} H^n(M;\mathbb{Z}) \xrightarrow{\mathbb{Z}} \mathbb{Z} \\
1 \xrightarrow{dV_N} \xrightarrow{(\deg F)dV_M} \xrightarrow{\deg F}
\end{array} \]

The isomorphisms between the \( n \)th cohomology groups and \( \mathbb{Z} \) are given by the specified orientations on \( M \) and \( N \). The following theorem of Hopf therefore generalizes the Gauss-Bonnet theorem:

Theorem 4 (Hopf 1925). Let \( M^{2k} \subset \mathbb{R}^{2k+1} \) be an even-dimensional codimension one smooth embedded hypersurface. If \( G : M \rightarrow S^{2k} \) is the Gauss map, then

\[ \deg G = \frac{1}{2} \chi(M), \]  

(5)

so

\[ \int_M KdV = \frac{m(S^{2k})}{2}\chi(M). \]  

(6)

Plainly we would now like to find a Gaussian curvature for arbitrary Riemannian manifolds, so that on even-dimensional manifolds \( M \) its integral was half the volume of the \( n \)th sphere times \( \chi(M) \). Is there such a thing?

\[ ^1 \text{Note } \chi(\Sigma) = 2 - 2g(\Sigma), \text{ where } g \text{ is the genus.} \]
The build-up

We must abandon, for the nonce, extrinsic notions. What do we have left? We have smooth functions, tangent spaces, vector fields, and a Riemannian metric, among other things. How now to define curvature?

To motivate what follows, let us recall the Frenet frame approach to the geometry of space curves $\gamma : [a, b] \to \mathbb{R}^3$. For simplicity, assume $\|\gamma'\| = 1$. We define the tangent vector field $T(t) = \gamma'(t)$, a unit vector field. Define the normal vector field $N(t)$ by $\gamma''(t) = T'(t) = N(t)\|\gamma''(t)\|$, another unit vector field. We write $\kappa(t) = \|\gamma''(t)\|$. Since $T$ is unitary, $T$ and $N$ are orthogonal. Let $B(t) = T(t) \times N(t)$. One can show $B'(t) = -\tau(t)N(t)$ for some function $\tau(t)$.

In fact, \[
\begin{cases}
T' = \kappa(t)N \\
N' = -\kappa(t)T + \tau(t)B \\
B' = -\tau(t)N
\end{cases}
\tag{7}
\]

**Theorem 5.** If $\bar{\gamma} : [a, b] \to \mathbb{R}^3$ has $\bar{\kappa}(t) = \kappa(t)$ and $\bar{\tau}(t) = \tau(t)$, then for some isometry $T : \mathbb{R}^3 \to \mathbb{R}^3$, $\bar{\gamma} = T \circ \gamma$.

**Proof.** Omitted. (cf. [10], p. 58.)

The 3-tuple $(T N B)$ is called the Frenet frame for $\gamma$. What this theorem says is that the derivatives of $(T N B)$ in terms of $(T N B)$ themselves completely determine the essential geometry of this embedding. This suggests that the derivatives of an orthonormal frame field on a Riemannian manifold reveals its geometry when expressed in terms of the original field. This is in fact so, as we will show below.

**Definition 6.** An oriented moving frame or frame field over an open set $U \subset M$ is an ordered $n$-tuple $(e_1, \ldots, e_n)$ of vector fields such that $\{e_1(p), \ldots, e_n(p)\}$ constitutes an oriented basis for $T_pM$ for every $p \in U$.

**Definition 7.** Every such moving frame has a dual moving coframe $\left( \begin{smallmatrix} \eta^1 \\ \vdots \\ \eta^n \end{smallmatrix} \right)$ determined by $\eta^i(e_j) = \delta^i_j$.

We regard a moving frame as a row vector $E$ of vector fields, and its dual coframe we regard as a column vector $H$ of 1-forms. We write $HE = I_n$.

**Proposition 8.** Given a moving frame $E$ and its dual coframe $H$, there is a unique skew-symmetric matrix $\omega$ of 1-forms satisfying $dH = -\omega \wedge H$, where $\wedge$ behaves like matrix multiplication:

\[
\begin{pmatrix}
\omega^1_1 & \cdots & \omega^1_n \\
\vdots & \ddots & \vdots \\
\omega^n_1 & \cdots & \omega^n_n
\end{pmatrix} \wedge \begin{pmatrix}
\eta^1 \\
\vdots \\
\eta^n
\end{pmatrix} = \begin{pmatrix}
\omega^1_1 \eta^1 + \cdots + \omega^1_n \eta^n \\
\vdots \\
\omega^n_1 \eta^1 + \cdots + \omega^n_n \eta^n
\end{pmatrix}.
\]
Definition 9. Such an $\omega$ is called the connection form for $E$.

Proof. As much as it pains me to say it, this proof illustrates the usefulness of indices. We first prove uniqueness: suppose $\omega$ to be such a matrix. Let $a^i_{jk} = \omega^i_j(e_k)$ and $b^i_{jk} = d\eta^i(e_j, e_k)$. So $a^i_{jk} = -a^i_{kj}$ and $b^i_{jk} = -b^i_{kj}$. Since $dH = -\omega \wedge H$, we have $d\eta^i = -\sum_k \omega^i_k \wedge \eta^k$ and

$$b^i_{jk} = -\sum_k \omega^i_k \wedge \eta^k(e_j, e_k)$$

$$= -\left( \sum_k \omega^i_k(e_j) \eta^k(e_k) - \omega^i_k(e_k) \eta^k(e_j) \right)$$

$$= \omega^i_j(e_k) - \omega^i_k(e_j) = a^i_{jk} - a^i_{kj}. $$

Permuting indices gives

$$b^i_{jk} - b^i_{kj} + b^i_{ki} = a^i_{jk} - a^i_{kj} - a^i_{ki} + a^i_{kj} + a^i_{kj} - a^i_{ki} = 2a^i_{jk}. $$

So the $a^i_{jk}$ are uniquely determined by the $b^i_{jk}$, which in turn depend solely upon $E$. This proves uniqueness.

For existence, just define $\omega^i_j(e_k) = a^i_{jk} = \frac{1}{2}(b^i_{jk} - b^i_{kj} + b^i_{ki})$. Then

$$\omega^i_j(e_k) = \frac{1}{2}(b^i_{jk} - b^i_{kj} + b^i_{ki}) = \frac{1}{2}(-b^i_{kj} + b^i_{kj} - b^i_{ki}) = -a^i_{jk}, $$

so $\omega$ is skew-symmetric. Furthermore, we can recover the relation

$$a^i_{jk} - a^i_{kj} = \frac{1}{2}(b^i_{jk} - b^i_{kj} - b^i_{ki} + b^i_{kj} + b^i_{kj} - b^i_{ki}) = b^i_{jk}; $$

therefore $d\eta^i(e_j, e_k) = \omega^i_j(e_k) - \omega^i_k(e_j) = -\sum_k \omega^i_k \wedge \eta^k(e_j, e_k). \quad \square$

$dH = -\omega \wedge H$ is called the first Cartan structural equation.

Now assume $M^n$ carries a Riemannian metric $\langle , \rangle$. We now consider orthonormal frame fields exclusively.

It can be shown that for any orthonormal frame field $E$ over $U \subset \mathbb{R}^n$, we have $d\omega = -\omega \wedge \omega$. However, for general Riemannian manifolds this is not so. The difference $d\omega + \omega \wedge \omega = \Omega$ measures the curvature of the manifold where $E$ is defined; $\Omega$ is called the curvature form for $E$.

Note that

$$\Omega^i_j = (d\omega)^i_j + (\omega \wedge \omega)^i_j$$

$$= d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j$$

$$= -d\omega^i_j - \sum_k \omega^k_i \wedge \omega^j_k$$

$$= -(d\omega)^i_j - \sum_k \omega^k_i \wedge \omega^k_j$$

$$= -(d\omega + \omega \wedge \omega)^i_j, $$
so $\Omega$ is skew-symmetric.

\[
\begin{align*}
\omega &= -\omega \land \omega + \Omega
\end{align*}
\]

is called the second Cartan structural equation.

Differentiating the Cartan structural equations yields the Bianchi identities

\[
\begin{align*}
\Omega \land H &= 0, \\
\omega \mathcal{L} \Omega - \Omega \mathcal{L} \omega &= 0.
\end{align*}
\]

Since we intend to construct a form intrinsic to the manifold using $\Omega$, it is paramount that we understand how it changes with a change of frame field. Since $\Omega$ is defined in terms of $\omega$, it behooves us to find how $\omega$ changes.

**Proposition 10.** Let $E, F$ be two frame fields over an open set $U \subset M$ related by $F_i = A E_i$, with $A : U \to SO(TU)$ a smooth orthogonal map. Let $\omega, \gamma$ be their respective connection forms.

Then $\gamma = A^{-1} dA + A^{-1} \omega A$.

This behavior of connection forms is the foundation for Cartan’s theory of connections, which can be used to define everything we’ve done here, generalizing Riemannian geometry. Interested readers are referred to Spivak’s second volume, from page 280 onward, and to Sharpe’s labor of love [11].

**Proof.** Let $\eta, \phi$ be the respective dual coframes of $E, F$.

Then $\phi'(V) = \eta'(A^{-1} V)$, and $\phi'(E_j) = \eta'(A^{-1} E_j) = (A^{-1})^j_i$; so $\Phi = A^{-1} H$.

Hence $d\Phi = dA^{-1} \land H + A^{-1} dH$. Now, $AA^{-1} = I$, so $dAA^{-1} = -A^{-1} dA^{-1}$, i.e. $dA^{-1} = -A^{-1} dAA^{-1}$. Therefore

\[
\begin{align*}
d\Phi &= -A^{-1} dAA^{-1} \land H + A^{-1} dH \\
&= A^{-1} dAA^{-1} \land H - A^{-1} \omega \land H = -(A^{-1} dAA^{-1} + A^{-1} \omega) \land H.
\end{align*}
\]

But also $d\Phi = -\gamma \land \Phi = -\gamma \land A^{-1} H = -(\gamma A^{-1}) \land H$. Now, setting $\bar{\gamma} = A^{-1} dA + A^{-1} \omega A$, plainly $d\Phi = -\bar{\gamma} \land \Phi$. On the other hand, since $A$ is orthogonal,

\[
\begin{align*}
\gamma(A^{-1} dA) &= \gamma dAA^{-1} = d' \omega A = d' \omega A = dA^{-1} A = -A^{-1} dA, \\
\gamma(A^{-1} \omega A) &= \gamma A' \omega A^{-1} = A^{-1} \omega A = A^{-1} \omega A,
\end{align*}
\]

so $\bar{\gamma}$ is also skew-symmetric. By our first proposition, $\gamma = \bar{\gamma}$. \hfill $\square$

**Proposition 11.** Let $E, F$ be frame fields over an open set $U \subset M$ related by $F_i = A E_i$, with $A : U \to SO(TU)$ a smooth orthogonal map. Let $\Omega, \Gamma$ be the respective curvature forms of $E$ and $F$.

Then $\Gamma = A^{-1} \Omega A$.

**Proof.**

\[
\begin{align*}
\gamma \land \gamma &= A^{-1} dA \land A^{-1} dA + A^{-1} dA \land A^{-1} \omega A + A^{-1} \omega \land dA + A^{-1} \omega \land \omega A \\
&= -dA^{-1} \land dA - dA^{-1} \land \omega A + A^{-1} \omega \land dA + A^{-1} \omega \land \omega A \\
d\gamma &= dA^{-1} \land dA + dA^{-1} \land \omega A + A^{-1} d\omega A - A^{-1} \omega \land dA \\
\Gamma &= d\gamma + \gamma \land \gamma = A^{-1} (d\omega + \omega \land \omega) A = A^{-1} \Omega A.
\end{align*}
\]
At this point we can recover a definition of Gaussian curvature for “abstract” Riemannian surfaces: any curvature form has the form \( \Omega = -K \left( \begin{array}{cc} 0 & \eta^1 \wedge \eta^2 \\ -\eta^1 \wedge \eta^2 & 0 \end{array} \right) \), and this is invariant under conjugation by orthogonal matrices. In fact, \( K \) is the Gaussian curvature as given above, and one can show (cf. [4]) already that von Dyck’s version of Gauss-Bonnet is true for arbitrary closed Riemannian surfaces using this definition.

For higher dimensions, we must do something extra. Let us once again consider a smooth, closed, orientable, even-dimensional, embedded codimension one hypersurface \( M^{2k} \subset \mathbb{R}^{2k+1} \). It may be shown that Gaussian curvature is still independent of embedding (cf. [12] 4, pp. 67–69); what’s more, tacking it on to the volume form gives the following curious formula (cf. [12] 5, p. 284):

\[
KdV_M = \frac{1}{(2k)!} \sum_{\sigma \in S_{2k}} \text{sgn}(\sigma) \Omega_\sigma^{(1)} \wedge \cdots \wedge \Omega_\sigma^{(2k-1)}. \tag{8}
\]

This is called the curvatura integra (cf. [3]); it was first called so by Hopf. In fact, this is a constant multiple of the Pfaffian of the curvature form matrix, as we now explain.

Let us denote by \( X \) the \( 2k \times 2k \) matrix of variable symbols \( X_{ij} \), and by abuse of notation the set of all such symbols therein.

For any commutative-ring-with-1 \( R \) (in our case, it is \( \Lambda^e(M) \), the \( R \)-algebra of differential forms of even degree, so that wedging is commutative), there is an obvious action of \( \mathbb{Z}[X] \) upon \( M_{2k}(R) \), viz. evaluation at respective matrix entries.

**Proposition 12.** For every \( k \), there exists one and, up to sign, only one polynomial \( \text{Pf} \in \mathbb{Z}[X] \) satisfying \( (\text{Pf}A)^2 = \det A \) for all skew-symmetric \( A \in M_{2k}(R) \). Furthermore, \( \text{Pf}(BA^tB) = \text{Pf}A \) for any matrix \( B \in M_{2k}(R) \).

**Proof.** We follow [8], p. 310. We add some variables \( Y_{ij} \) to our polynomial ring \( P = \mathbb{Z}[X,Y] \) to accommodate \( B \). Over the quotient field of \( P \) we may find a matrix \( C \) such that

\[
CX^tC = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & -1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & 0 & \cdots & -1 & 0
\end{pmatrix}
\]

by echelon row-reduction on both sides of \( X \). So \( \det C \det X \det^tC = 1 \), i.e. for some matrix \( C \) over the quotient field of \( P \), \( (\det C)^{-2} = \det X \). That is, \( \det \) has a square root in the quotient field of \( P \). Since \( P \) is a unique factorization domain, this implies that \( \det \) already had a square root Pf in \( P \).

With regards to the second formula, note \( \det(BA^tB) = \det B^2 \det A \), so \( \text{Pf}(BA^tB) = \pm \det B \text{Pf} A \). Letting \( B = I \), plainly it is + rather than -.
Definition 13. Pf is called the Pfaffian.

It turns out that in our situation, \( \text{Pf} X = \frac{1}{2^k k!} \sum_{\sigma \in S_{2k}} \text{sgn}(\sigma) x^{\sigma(1)} x^{\sigma(2)} \cdots x^{\sigma(2k)} \), so (as we mentioned above), in the case of embedded submanifolds,

\[
\frac{2^k k!}{(2k)!} \text{Pf} \Omega = KdV. \tag{9}
\]

Since Pf is invariant under similarity, we see that this definition of Gaussian curvature on embedded submanifolds is independent of embedding, the Theorema Egregium writ large! For arbitrary Riemannian manifolds, we take this to be our definition of Gaussian curvature by analogy. We hope that as in the embedded case, we still have (6). This is so, and it is a theorem of Chern\(^2\).

Chern’s proof\(^3\)

First, Chern’s proof uses the Poincaré-Hopf index theorem in an essential way. The phrase “Poincaré-Hopf index theorem,” so far as I can tell, refers to a triple of results relating Euler characteristic, intersection theory, and vector field indices. The Euler characteristic of a smooth manifold is just the alternating sum of the ranks of its homology groups. The Euler number is the intersection number \( I(\Delta, \Delta) \) of the diagonal \( \Delta = \{(p, p) | p \in M\} \subset M \times M \) with itself (cf. [5] p. 116, also [1], p. 382); for the definition of intersection number, I recommend Guillemin and Pollack’s book. The index of a nonzero unit vector field \( V : U \smallsetminus \{p\} \to TM \) at \( p \in U \) is (informally) given by restricting \( V \) to a small sphere \( S \) about \( p \) in \( U \), and taking the degree of the corresponding map \( V|_S : S \to S^n \), writing \( \text{deg} V|_S = \text{ind}_p V \).

Then

\[
\sum_{i=1}^{n} \text{ind}_{p_i} V = \chi(M). \tag{Cf. [6], pp. 29–36, also [7], pp. 35–41.}
\]

If \( V : M \smallsetminus \{p_1, \ldots, p_n\} \to UTM \) is a unit vector field, then \( \sum_{i=1}^{n} \text{ind}_{p_i} V \) is the Euler number. (Cf. [5], pp. 134–137.)

\(^2\)Chern gave the first intrinsic proof, but Allendoerfer and Weil proved it first.

\(^3\)Until further notice, I will only provide here a short summary of what is involved in his proof, and refer the interested reader to Chern’s original papers and Wu’s summary. (Also, Flanders essentially follows Chern’s proof for the instructive two-dimensional case in [4].)

\(^4\)This isn’t quite right since the map \( V|_S \) involves a choice of isometry \( \{\text{unit vectors in } T_p M\} \leftrightarrow S^n \), but this gives the general idea of index.
With that caveat, the idea of Chern’s proof is the following: we consider the pullback of the “curvatura integra” $KdV$ to the unit tangent bundle. That is, if $\pi : UT M \to M$ is the unit tangent bundle (where $\pi$ takes a unit vector to the point at which it is tangent to the manifold), we consider $\pi^*(KdV)$, or equivalently some multiple of $\pi^*(\text{Pf } \Omega)$.

**Theorem 15** (Chern 1944). There is an $(n-1)$-form $\Pi$ on $UT M$ so that $d\Pi = \pi^*(KdV)$, and so that $\Pi$ restricts on the fiber $\pi^{-1}(p) \approx S^{k-1}$ to $(-1)^k(2k-1)!$ times the volume form $\sigma_{2k-1}$.

Equivalently, setting $\mathcal{U} = \frac{2}{m(2k+1)}KdV$, there is an $(2k-1)$-form $\mathbb{A}$ on $UT M$ so that $d\mathbb{A} = \mathcal{U}$ and such that $\mathbb{A}$ restricts on the fiber $\pi^{-1}(p)$ to $\frac{(k-1)!}{2\pi^k}m(S^{2k-1}) = \frac{\chi(M)}{2}$.

Chern’s proof of the above theorem is involved, but compact. It involves defining two auxiliary sequences of invariant differential forms and setting up a recurrence relation between these. An exposition of this core of Chern’s proof is here omitted until further notice.

With this theorem, the proof of the general Gauss-Bonnet equation (6) follows exceedingly elegantly. Let $V : M \setminus p \to TM$ be a unit vector field with one singularity $p$. Then since $V$ is a section of the unit tangent bundle (i.e. $\pi \circ V = 1_{M \setminus p}$ is the identity map), we may write

$$\int_M \mathcal{U} = \int_{M \setminus p} \mathcal{U} = \int_{M \setminus p} V^* \circ \pi^* \mathcal{U} = \int_{M \setminus p} V^* d\Pi = \text{ind}_p V \int_{V(M \setminus p)} d\Pi = \chi(M) \int_{\partial V(M \setminus p)} \Pi = \chi(M) \int_{\pi^{-1}(p)} \Pi = \chi(M) \frac{1}{2} \frac{(k-1)!}{\pi^k} m(S^{2k-1}) = \frac{\chi(M)}{2},$$

which was what we wanted.

In conclusion, we should mention that this theorem has a (much) more conceptual proof using characteristic classes, and in fact it was this theorem of Chern’s, in part, that inspired the modern theory of characteristic classes. In particular, the form which we have denoted by $\mathcal{U}$ above is more properly referred to as the Euler class of the tangent bundle. The relative lack of computation in a proof of the Chern-Gauss-Bonnet theorem using this theory is described as a “proof by magic” in Spivak’s text. Indeed, the computation is all there, but it is taken up and hidden within Grassmann bundles, so that proof!—the factorials all seem to vanish, and we’re left with a “clean” proof, such as that given in Spivak or Milnor and Stasheff. Perhaps we should distinguish between the “legerdemain” that is characteristic of such notational cleanups (Stokes’s theorem is another example of this phenomenon), and the truly magical, inspired ideas of such wizards as Chern and Gauss.

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5 The singularity will most likely be degenerate in the sense of Morse, which is to say that if we mollified it by locally smoothing the field to be zero there, the Hessian of the zero would be singular, i.e. have zero determinant.

6 In passing, we should warn the reader that our notation differs from Chern’s in significant and confusing ways.

7 Morita’s proof is also typically lucid, and recommended.
References


