Some tips on the comparison test

April 13, 2011

The comparison test as we have treated it deals with positive functions. It basically says that:

1. If we suspect \( \int_a^b f(x) \, dx \) converges, then we wish to find a nice function \( g \) so that \( g(x) \geq f(x) \) and \( \int_a^b g(x) \, dx \) converges. The convergence of \( \int_a^b g(x) \, dx \) will imply the convergence of \( \int_a^b f(x) \, dx \) by comparison.

2. If we suspect instead that \( \int_a^b f(x) \, dx \) diverges, then we wish to find a nice function \( h \) so that \( h(x) \leq f(x) \) and \( \int_a^b h(x) \, dx \) diverges. The divergence of \( \int_a^b h(x) \, dx \) will imply the divergence of \( \int_a^b f(x) \, dx \) by comparison.

We should point out two similar but useless situations:

1. If \( g(x) \geq f(x) \) and \( \int_a^b g(x) \, dx \) diverges, then so what; who cares! This says nothing about \( \int_a^b f(x) \, dx \).

2. If \( h(x) \leq f(x) \) and \( \int_a^b h(x) \, dx \) converges, then so what; who cares! This says nothing about \( \int_a^b f(x) \, dx \).

Now, some of you have pointed out rightly that it is not always clear whether to suspect convergence or divergence and after deciding this it is still not clear how to find a nice bound (i.e. a nice function \( g \) or \( h \)). In formulating your guesses it is very important to remember the following two things:

**Theorem 1** For \( a > 0 \),

\[
\int_a^\infty \frac{1}{x^p} \, dx = \begin{cases} 
\frac{1}{p-1}, & p > 1, \\
\infty, & \text{else}.
\end{cases}
\]

In particular, this integral converges for \( p > 1 \) and diverges for \( p \leq 1 \).

**Theorem 2** For \( a > 0 \),

\[
\int_0^a \frac{1}{x^p} \, dx = \begin{cases} 
\frac{1}{1-p}, & p < 1, \\
\infty, & \text{else}.
\end{cases}
\]

In particular, this converges for \( p < 1 \) and diverges for \( p \geq 1 \).
These two theorems illustrate the two issues that arise in evaluating improper integrals: poles (infinite discontinuities) and limits at infinity.

Some other important things to remember:

1. The function \( e^x \) grows more quickly than any power of \( x \) as \( x \to \infty \). That is to say, for any power \( \alpha \), \( \lim_{x \to \infty} \frac{e^x}{x^\alpha} = 0 \).

2. The function \( \log x \) grows more slowly than any positive power of \( x \) as \( x \to \infty \). That is to say, for any positive power \( \alpha > 0 \), \( \lim_{x \to \infty} \frac{\log x}{x^\alpha} = 0 \).

3. For positive \( x \), \( \frac{1}{x^{p+q}} \leq \frac{1}{x^q} \).

4. If \( x > 0 \) and \( p \leq q \), then \( \frac{1}{x^{p+q}} \leq \frac{1}{2x^q} \).

Let me illustrate a slightly general approach to the problem of convergence of the integral of a function that looks something like the following. This particular function is artificially and deliberately terrifying; I hope that taming this function’s integral will encourage you to tame other, less frightening integrals.

Let 
\[
f(x) = \frac{2x^{7/2} + \pi x - 30 + \frac{1}{\sqrt{x}} + \frac{5}{x^7}}{x^4 + 3 - \frac{1}{x^4} - \frac{1}{x^7}}.
\]

We wish to determine the convergence of the integral
\[
\int_2^\infty f(x) \, dx.
\]

Step 1 Don’t panic. (This is easier said than done, unfortunately.)

Step 2 Find the highest power in the numerator, and the highest power in the denominator, and subtract the former from the latter to get some number \( d \). In this instance, the highest power in the numerator is \( \frac{7}{2} \), and the highest power in the denominator is \( 4 \). So the difference I am referring to in this instance is \( 4 - \frac{7}{2} = \frac{1}{2} = d \).

Step 3 Determine the convergence of the integral of \( \int_1^\infty \frac{1}{x^d} \, dx \) using Theorem 1. In this instance, \( \int_1^\infty \frac{1}{x^{1/2}} \, dx \) does not converge.

Step 4a If \( \int_1^\infty \frac{1}{x^d} \, dx \) diverges, then our original integral diverges. In this instance, \( \int_2^\infty f(x) \, dx \) diverges.

Step 4b If, instead, \( \int_1^\infty \frac{1}{x^d} \, dx \) converges, and if our original integral doesn’t have any infinite discontinuities in it, then our original integral converges.

Roughly, this means that the behavior of \( f \) at infinity is basically that of \( \frac{1}{x^d} \). Similarly, if one takes the difference \( c \) of the smallest power of \( x \) in the denominator from the smallest power of \( x \) in the numerator, and if the only pole of \( f \) on \([0, A]\) is 0, then \( \int_0^A f(x) \, dx \) converges precisely when \( \int_0^A \frac{1}{x^d} \, dx \).