Determining hyperbolicity of compact orientable 3-manifolds with torus boundary

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Abstract

Thurston's hyperbolization theorem for Haken manifolds and normal surface theory yield an algorithm to determine whether or not a compact orientable 3-manifold with nonempty boundary consisting of tori admits a complete finite-volume hyperbolic metric on its interior.

A conjecture of Gabai, Meyerhoff, and Milley reduces to a computation using this algorithm.

1 Introduction

The work of Jørgensen, Thurston, and Gromov in the late '70s showed that the set of volumes of orientable hyperbolic 3-manifolds has order type ω^ω. Cao and Meyerhoff in 2001 showed that the first limit point is the volume of the figure eight knot complement. Agol in 2010 showed that the first limit point of limit points is the volume of the Whitehead link complement. Most significantly for this paper, Gabai, Meyerhoff, and Milley in 2009 showed that the smallest, closed, orientable hyperbolic 3-manifold is the Weeks-Matveev-Fomenko manifold.

The proof of the last result required distinguishing hyperbolic 3-manifolds from non-hyperbolic 3-manifolds in a large list of 3-manifolds; this was carried out in [7]. The method of proof was to see whether SnapPea's *canonize* procedure succeeded or not; identify the successes as census manifolds; and then examine the fundamental groups of the 66 remaining manifolds by hand. This method made the analysis of non-hyperbolic Mom-4 manifolds, of which there are 762 combinatorial types, prohibitively time-consuming.

The algorithm presented here determines whether or not a compact 3-manifold admits a complete finite-volume hyperbolic metric, i.e. is *hyperbolic*, assuming the manifold in question has nonempty boundary consisting of tori.

The Mom-4s have such boundaries. The current implementation of this algorithm using Regina (see [3]) classifies them, yielding the following result.

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m125	m129	m202	m203	m292	m295	m328
m329	m357	m359	m366	m367	m388	m391
m412	s441	s443	s503	s506	s549	s568
s569	s576	s577	s578	s579	s596	s602
s621	s622	s638	s647	s661	s774	s776
s780	s782	s785	s831	s843	s859	s864
s880	s883	s887	s895	s898	s906	s910
s913	s914	s930	s937	s940	s941	s948
s959	t10281	t10700	t11166	t11710	t12039	t12044
t12046	t12047	t12048	t12049	t12052	t12053	t12054
t12055	t12057	t12060	t12064	t12065	t12066	t12067
t12143	t12244	t12412	t12477	t12479	t12485	t12487
t12492	t12493	t12496	t12795	t12840	t12841	t12842
v2124	v2208	v2531	v2533	v2644	v2648	v2652
v2731	v2732	v2788	v2892	v2942	v2943	v2945
v3039	v3108	v3127	v3140	v3211	v3222	v3223
v3224	v3225	v3227	v3292	v3294	v3376	v3379
v3380	v3383	v3384	v3385	v3393	v3396	v3426
v3429	v3450	v3456	v3468	v3497	v3501	v3506
v3507	v3518	v3527	v3544	v3546		

Table 1: Names of hyperbolic Mom-4s.

Theorem 1. *Table 1 constitutes the complete list of hyperbolic Mom-4s.*

Proof. Put the Python modules `unhyp` and `mom` in one directory also containing the data from [4], which here is called `mm4.txt`. Then from a `bash` prompt, run

```
python mom.py mm4.txt > hyp4s
```

One finds that `hyp4s` has 138 lines, each of which is a distinct name of a cusped manifold on a census, either the original SnapPea census or Thistlethwaite's more recent census of manifolds with eight tetrahedra. Therefore, Conjecture 5.3 from [4] is correct. □

Of course, now we should discuss what is in these modules.

Remark 2. The author would like to thank Tao Li for helpful discussions about normal surface theory and Seifert fiberings, and especially thank Dave Futer for pointing out an error in a previous version of the paper.

2 Background

Conventions. All manifolds herein are assumed to be compact and piecewise-linear. All maps between these are assumed to be piecewise-linear and proper (that is, such that the preimage of compacta are again compacta). In particular, all homeomorphisms are piecewise-linear with piecewise-linear inverses. (End Conventions.)

Thurston's hyperbolicity theorem for Haken manifolds merits a succinct formulation. Shoving some complications from the original theorem into definitions and restricting attention to manifolds with nonempty torus boundary yields

Theorem 3. *Let M be a compact orientable 3-manifold with nonempty boundary consisting of tori.*

M is hyperbolic with finite volume if and only if M has no faults.

The above uses the following definitions.

Definition 4. A manifold is *hyperbolic* when its interior admits a complete hyperbolic metric—a complete Riemannian metric of constant negative curvature.

Definition 5. Let s be an embedding of a manifold into a connected manifold M. By abuse of notation, also let s denote the image of s in M. Suppose s has codimension 1. Pick a metric on M compatible with its p.l. structure, and let M' be the path-metric completion of $M \setminus s$.

When M' is disconnected, *s separates M*.

When M' has two connected components N, N', *s cuts off N from M*, or, if M is understood from context, *s cuts off N*.

If N is homeomorphic to some common 3-manifold X, *s cuts off an X*; if, in addition, N' is not homeomorphic to X, s cuts off *one* X.

Definition 6. A properly embedded surface s in an orientable 3-manifold M is a *fault* when $\chi(s) \geq 0$ and it satisfies one of the following:

- s is nonorientable.
- s is a sphere which does not cut off a 3-ball.
- s is a disc which does not off one 3-ball.
- s is a torus which does not cut off a $T^2 \times I$, and does not cut off a ∂-compressible manifold.
- s is an annulus which does not cut off a 3-ball, and does not cut off one solid torus.

Sketch of Thm. 3's proof. This is a corollary of common knowledge surrounding Thurston's hyperbolization theorem for Haken manifolds. Specifically, it's commonly known that an irreducible, ∂-incompressible, geometrically atoroidal 3-manifold with nonempty boundary consisting of tori is either hyperbolic or Seifert-fibered, where "hyperbolic" means "admits a complete hyperbolic metric." All Seifert-fibered spaces with at least two boundary components admit essential tori, which are faults. A Seifert-fibered space with one boundary component admits no essential tori. But it still admits an annulus fault, namely a vertical fiber over an arc separating the cone points of the base orbifold, which is a disc with two cone points. Hence all Seifert-fibered spaces with nonempty boundary admit faults.

Consequently, a compact orientable 3-manifold with nonempty boundary consisting of tori which admits no faults is irreducible, ∂-incompressible, Haken, and geometrically atoroidal, and it admits no annulus faults. So it must be hyperbolic.

In fact, Thurston proved something more, namely that unless this manifold is $T^2 \times I$, then its metric has finite volume. Now, $T^2 \times I$ admits faults—non-separating annuli, in fact. Since we assumed the manifold had no faults, its metric must have finite volume.

Conversely, hyperbolic 3-manifolds of finite volume admit no orientable faults—they have no essential spheres, no compressing discs, no incompressible tori which aren't ∂-parallel, and no annuli which are both incompressible and ∂-incompressible. Finally, orientable hyperbolic 3-manifolds of finite volume don't admit any faults at all, since they admit no properly embedded nonorientable surfaces of nonnegative Euler characteristic. □

Having finished this first reformulation, we note the following theorems from normal surface theory.

Theorem 7. *Let T ideally triangulate a compact orientable 3-manifold M. Then M has a closed fault precisely when T has a fundamental normal fault.*

Theorem 8. *Let T finitely triangulate an irreducible, ∂-incompressible, geometrically atoroidal 3-manifold M with nonempty boundary consisting of tori.*

1. *M has a fault if and only if T has a vertex Q-normal annulus fault.*
2. *If M has at least two boundary components, then M has a fault if and only if T has a non-separating vertex Q-normal annulus fault.*

The last section contains proofs of these statements.

The reader need know little about the definition of the adjectives, except the following: for any ideal triangulation, its fundamental normal surfaces are a finite computable set, and likewise the vertex Q-normal surfaces of a finite triangulation form a finite computable set.

In particular, one may put all the fundamental normals or vertex Q-normals into a finite list and iterate a test over this list—say a test for faults—and guarantee termination for this iteration.

Theorems 3, 7, and 8 together yield the following useful results amenable to computer implementation.

Corollary 9. *Let M be a compact orientable 3-manifold with nonempty boundary consisting of tori.*

Let T, T' triangulate M ideally and finitely, respectively.

M is hyperbolic precisely when T has no fundamental normal closed fault, T' has no disc fault, and T' has no vertex Q-normal annulus fault.

Corollary 10. *The last condition in Corollary 9 can be relaxed to having no non-separating vertex Q-normal annulus fault in case* $|\partial M| \geq 2$.

Therefore, assuming T is an ideal triangulation of a compact orientable 3-manifold M with nonempty boundary consisting of tori,

```
l := list of fundamental normal surfaces in T
for surf in l:
  if surf is fault:
    return False
T' := truncation of T to finite triangulation
if T' has a compressing disc:
  return False
l' := list of vertex Q-normal surfaces in T'
for annulus in l':
  if M has at least two boundary tori:
    if annulus is non-separating:
      return False
  else:
    if annulus is fault:
      return False
else:
  return True
```

describes an algorithm determining whether or not M is hyperbolic.

Of course, this algorithm depends upon

- enumerating fundamental normal surfaces of ideal triangulations;
- truncating ideal triangulations into finite triangulations;
- the predicate "has a compressing disc";
- enumerating vertex Q-normal surfaces of finite triangulations;
- the predicate "is non-separating annulus"; and
- the predicate "is fault."

All but the last two are already described in the existing literature and implemented conveniently in Regina.

The relevant tests and algorithms for detecting non-separating annuli are in Regina already—calculating Euler characteristic, cutting along a surface, and determining whether or not a manifold is connected.

The relevant tests for faultiness (all but the last of which are in Regina) are

- "is a 3-ball"
- "is ∂-compressible"
- "is a solid torus", and

- “is $T^2 \times I$”.

We can notice first that admitting a non-separating annulus is a necessary condition for being $T^2 \times I$. We note that a further necessary condition for being $T^2 \times I$ is that splitting along any such annulus is a solid torus. Now, if a 3-manifold M split along a non-separating annulus is a solid torus, then M is a Seifert fibering with base orbifold an annulus with at most a single cone point, i.e. $M = M(\pm 0, 2; r)$ for some $r \in \mathbb{Q}$. Recall the following results about Seifert fiberings:

Proposition 11 ([6], 2.1)**.** *Every orientable Seifert fibering is isomorphic to one of the models $M(\pm g, b; s_1, \ldots, s_k)$. Any two Seifert fiberings with the same $\pm g$ and b are isomorphic when their multisets of slopes are equal modulo 1 after removing integers, assuming $b > 0$.*

Theorem 12 ([6], 2.3)**.** *Orientable manifolds admitting Seifert fiberings have unique such fiberings up to isomorphism, except for $M(0, 1; s)$ for all $s \in \mathbb{Q}$ (the solid torus), $M(0, 1; 1/2, 1/2) = M(-1, 1;)$ (not the solid torus), and three others without boundary.*

Proposition 13. *Among manifolds of the form $M(\pm 0, 2; r)$, only $T^2 \times I$ has all Dehn fillings being solid tori.*

Proof. Plainly $T^2 \times I$ has this property.

Suppose $M(0, 2; r)$ is not $T^2 \times I$. Then by Proposition 11 and Theorem 12, $r \notin \mathbb{Z}$. We wish to show that $M(0, 2; r)$ admits some Dehn filling which is not a solid torus. Let s, s' be two slopes differing mod 1. Then $M(0, 1; r, s)$ and $M(0, 1; r, s')$ are Dehn fillings of $M(0, 2; r)$. They are not homeomorphic, by Theorem 12 and the fact that $r \notin \mathbb{Z}$. So one of them is not a solid torus. □

It is quite easy to compute slopes differing mod 1 after simplifying the cusps' induced triangulations.

Proposition 14. *In a triangulation of the torus T^2 by one vertex, three edges, and two faces, for any nontrivial element g of $H_1(T^2)$, the edges represent homology classes not all equivalent mod g.*

Proof. Suppose $v, w, x \in H_1(T^2)$ and $v + w = x$. Let $\equiv$ denote equivalence in $H_1(T^2) \mod g$. Then

$$
\begin{aligned}
& v \equiv x \\
\Rightarrow \quad & \\
& v + w \equiv x + w \\
= \quad & \{v + w = x\} \\
& x \equiv x + w \\
= \quad & \\
& 0 \equiv w;
\end{aligned}
$$

assuming $v \equiv w$ then implies v and x also are $0 \mod g$. Therefore they are all multiples of g. But $H_1(T^2)$ is not cyclic. So v, w, x cannot generate $H_1(T^2)$.

Now, one may pick homology classes v, w, x representing the three edges such that $v + w = x$. These generate $H_1(T^2)$. Therefore, by the above argument, they cannot satisfy $v \equiv w \equiv x \mod g$ for any element g. □

Corollary 15. *The following pseudocode describes an algorithm determining whether or not a compact, orientable, 3-manifold M with nonempty boundary consisting of tori is $T^2 \times I$:*

```
if M splits along no annulus into a solid torus:
  return False
let D be M's triangulation
let T be a boundary component of M
let tr(T,D) be the triangulation on T induced from D
change D so tr(T,D) has 2 faces, 3 edges, and 1 vertex
if M filled along one of the 3 edges' slopes is not a solid torus:
  return False
else:
  return True
```

Proof. Suppose M is $T^2 \times I$. Then the first `if`-statement doesn't activate, for M splits along an annulus into a solid torus. Also, M filled along any edge's slope whatever is a solid torus, so the second `if`-statement doesn't activate. So the algorithm returns `True`.

Suppose instead that M is not $T^2 \times I$. If M splits along no non-separating annulus into a solid torus, then the algorithm correctly returns `False`. Otherwise, M does so split, and therefore $M = M(0, 2; r)$ for some $r \in \mathbb{Q} \setminus \mathbb{Z}$. The algorithm then establishes that M's triangulation induces a minimal triangulation on the boundary component T. By Proposition 14, the edges represent at least two different slopes modulo 1. Therefore, by Proposition 11 and Theorem 12, the Dehn fillings of M along these slopes are not all homeomorphic. In particular, one of them is not a solid torus. Therefore, the `if`-statement activates, and the algorithm correctly returns `False`. □

It remains to describe

- splitting along a non-separating annulus into a solid torus,
- changing a triangulation to induce a minimal triangulation on a cusp, and
- filling along a slope in a simplified cusp.

Proposition 16. *The following pseudocode describes an algorithm implementing the first item:*

```
for every vertex Q-normal surface s in M:
  if s is a non-separating annulus:
    if M splits along s into a solid torus:
```

```
    return True
return False
```

Proof. Suppose M doesn't split along a non-separating annulus into a solid torus. Then not both of the `if`-statements can activate, so the `for` loop ends without returning, and so the algorithm correctly returns `False`.

On the other hand, if M does split along a non-separating annulus into a solid torus, then M is of the form $M(0, 2; r)$. By Lemma 25, every finite triangulation of such a manifold admits a non-separating Q-vertex annulus. Hence the `if`-statements eventually activate, and the algorithm correctly returns `True`. □

Now for the next item, simplifying cusps. One may find a nice algorithm in SnapPea for doing this, a special, simpler case of which is presented here. We use the following terminology.

Definition 17. First, suppose M is finitely triangulated. Let T, T' be boundary triangles adjacent along an edge e. Orient e so that T lies to its left and T' to its right. Let Δ be a fresh tetrahedron, and let τ, τ' be boundary triangles of Δ adjacent along an edge η. Orient η so that τ lies to its left and τ' to its right. Without changing M's topology we may glue Δ to T by gluing η to e, τ to T' and τ' to T. This is called a *two-two* move.

In the above definition, the edge η' opposite η in Δ becomes a boundary edge of the new finite triangulation.

Definition 18. We say e is *embedded* when its vertices are distinct. We say e is *coembedded* when η' as defined above is embedded. Equivalently, e is coembedded when the vertices in T, T' opposite e are distinct.

Given a boundary edge e between two boundary triangles T and T', one may glue T to T' and e to itself via a valid, orientation-reversing map from T to T'. This identification we call "folding along e". (Weeks, in the SnapPea source code, calls this a "close-the-book" move.) This gluing will change the topology of M when the vertices opposite e in T and T' are the same vertex. Conversely, when these vertices are distinct, the folding preserves the topology. In other words, folding along e preserves topology if and only if e is coembedded.

Notice that folding along a coembedded edge decreases the number of boundary triangles, and performing a two-two move on an embedded edge produces a coembedded edge and preserves the number of boundary triangles. Therefore, the following while-loops terminate, using number of boundary triangles as a variant function:

```
while there's an embedded boundary edge e:
  do a two-two move on e
  while there's a coembedded boundary edge f:
    fold along f
```

The obvious postcondition of the outer while loop is that there is no embedded boundary edge. Since the boundary is still triangulated, this is equivalent to each boundary component having only one vertex on it. Since each boundary component is a torus, $V - E + F = 0$. Now, $V = 1$, and since the cellulation is a triangulation, $3 * F = 2 * E$.

$$
\begin{aligned}
1 - E + F &= 0 \\
2 - 2 * E + 2 * F &= 0 \\
2 - 3 * F + 2 * F &= 0 \\
2 - F &= 0 \\
2 &= F,
\end{aligned}
$$

and there are only two triangles and three edges.

The routine in SnapPea is more complicated because, rather than filling in a cusp any old way, SnapPea wants to make sure the filling compresses some given slope in the cusp.

In conclusion,

Proposition 19. *The following pseudocode changes a finite triangulation D with boundary consisting of tori so that D induces a minimal triangulation on every boundary component:*

```
while D has an embedded boundary edge e:
  do a two-two move on e
  while D has a coembedded boundary edge f:
    fold along f
```

Proof. See above discussion. □

Finally,

Proposition 20. *Assuming a triangulation D has a torus boundary component T and induces a minimal triangulation thereon, the following pseudocode determines whether folding along one of the edges in T yields a solid torus:*

```
for each edge e in T:
  let N be D folded along e
  if N is a solid torus:
    return True
return False
```

Proof. Omitted. □

This concludes the present sketch of an algorithm to determine hyperbolicity of a compact, orientable 3-manifold with nonempty boundary consisting of tori. Both literate and raw implementations of this algorithm as a Regina-Python module `unhyp` reside at [5]. Also available at [5] is a Regina Python module `mom` for interpreting Milley's data as manifolds in Regina.

3 Normal Surface Corollaries

Proof of Thm. 7. This is just breaking down the definition of closed fault and using the following classical theorems of normal surface theory. The following are such theorems from [9], with implications trivially spruced up to equivalences. We also abbreviate "fundamental normal" to "fundamental."

Theorem 21 ([9], 4.1.12)**.** *Let M be an orientable ideally triangulated 3-manifold.*

- *M has a projective plane if and only if M contains a fundamental projective plane.*
- *M has an essential[1] sphere if and only if M contains either a fundamental sphere or a fundamental projective plane.*

So M has a closed fault of $\chi > 0$ when it has a fundamental closed such fault.

Now for closed faults of $\chi = 0$. The next theorem is more complicated than we need.

Theorem 22 ([9], Lemma 6.4.7)**.** *Let (M, Γ) be an orientable irreducible triangulated 3-manifold. (M, Γ) contains an essential torus if and only if (M, Γ) contains either a fundamental essential torus or a normal essential torus which double covers a fundamental Klein bottle.*

We can ignore the Γ since this is a theorem about closed surfaces. (Those not satisfied with this assertion may take comfort in the remark on p. 234 of [9] after the definition of "essential.") That is to say,

Corollary 23. *Let M be an orientable irreducible triangulated 3-manifold. M contains an essential torus when M contains either a fundamental essential torus or a normal essential torus which double covers a fundamental Klein bottle.*

Now M contains a closed fault of $\chi = 0$ when either it contains a T^2 fault or it contains an embedded K^2. The former case is equivalent to M containing a fundamental T^2 or an essential double cover of a fundamental normal K^2. The latter case is not much more subtle. Just note that Lemma 6.4.7 is proved, in part, by showing that a minimal-edge-degree injective K^2 is fundamental. Such a K^2 exists if and only if M admits an injective K^2.

Therefore, M contains a closed fault of $\chi = 0$ if and only if it contains a fundamental such fault. Consequently, M contains a closed fault if and only if it contains a fundamental closed fault. □

Regina takes care of finding disc faults for us already, so it remains to examine annulus faults. Before the proof proper, let's explain why we use Q-normal coordinates and non-separating annuli.

One may certainly derive theorems for annulus faults as above, and indeed Matveev does. However, to use these theorems on an ideal triangulation to find

[1] Here, "essential" means "does not bound a 3-ball"; see [9], Remark 3.3.3.

annulus faults, one must truncate the ideal triangulation to get a finite triangulation first. Even after simplifying the result, one gets many more tetrahedra in the triangulation, and the number of fundamental surfaces increases greatly.

In the case of a finite triangulation, there is an alternative to usual normal coordinates, viz. Q-normal or Q-normal coordinates. The space of Q-normal surfaces is smaller than that of normal surfaces; it excludes obviously trivial surfaces. So we use that here, because it is faster than fundamental normal surface enumeration.

Now, one may test whether an annulus is a fault or not. However, if the annulus does not separate, this entails "isSolidTorus" and "isBall" tests, which are costly. So it would be great if we could skip them altogether. And, as we shall show, indeed we can, if $|\partial M| \geq 2$.

Proof of Thm. 8. Let M be a compact orientable 3-manifold with nonempty boundary consisting of tori.

By Thurston's hyperbolization theorem, if M is irreducible, ∂-irreducible, and geometrically atoroidal, then it admits a homogeneous geometry. This is either hyperbolic, Euclidean, spherical, Seifert-type, or solve. ∂M, and hence M itself, has infinite π_1, so the geometry is not spherical. By theorem 2.11 of [2], solvegeometric structures either fiber over S^1 with T^2 fiber, which fibers are faults, or are line bundles over bases T^2, K^2, an annulus, a Möbius strip, or a disc (Bonahon calls this disc a *plane*), which bases are faults. M has no faults, so it doesn't admit solvegeometry. Finally (for now), note that if M admits a Euclidean structure, then by theorem 2.7 of [2], either M is Seifert-fibered, or is an orientable disc bundle over S^1, which is a solid torus (which admits a Seifert fibering), or is a ball. M is not a ball, since it has torus boundary. In conclusion, then, either M is hyperbolic or it admits a Seifert fibering.

Let M have a Seifert fibering with base orbifold Σ. Since by assumption M admits no torus faults (this is what geometric atoroidality means), it admits in particular no vertical torus faults. Hence there are no essential loops in Σ. Therefore Σ is a sphere with c cone points and b boundary components such that $b + c \leq 3$. We say M is *medium* Seifert-fibered, or is a medium Seifert fibering.

We have two consequents to prove.

1. Without further assumptions on M we are to prove that M admits a vertex Q-normal annulus fault. It will plainly suffice to prove the following lemma.

   Lemma 24. *Let M be a medium Seifert fibering with nonempty boundary. Every finite triangulation of M admits a vertex Q-normal annulus fault.*

   Then either M is hyperbolic, in which case it admits no annulus faults whatever, or M is medium Seifert-fibered with nonempty boundary, in which case the lemma applies, and T admits a Q-vertex annulus fault.

2. In case $|\partial M| \geq 2$, it suffices as above to prove a similar lemma, viz.

Lemma 25. *Let M be a medium Seifert fibering with at least two boundary tori.*

Every finite triangulation of M admits a non-separating vertex Q-normal annulus fault.

□

To prove these lemmata, we should at least recall what *vertex* and *Q-normal* all mean. The reader unfamiliar with normal surface theory should now go read section 1 of [8]. Let T finitely triangulate a compact orientable 3-manifold M. Now, Haken's normal surface equations are homogeneous and linear, with some linear inequalities tacked on at the end. Therefore the solution space can be projectivized, which in [8] is denoted by $\mathcal{P}$ or $\mathcal{P}_{\mathcal{T}}$ when being explicit about the given triangulation. Because the equations and inequalities are linear, this solution space is a convex polytope in $PR^{7\cdot t-1}$, where t is the number of tetrahedra in T, and R is one's field of choice, here $\mathbb{Q}$. If S is an embedded normal surface, then it projects to a point on this polytope. The smallest face of $\mathcal{P}$ containing S is called the *carrier* of S, written $\mathcal{C}(S)$. The vertices of $\mathcal{P}$ may or may not be projections of embedded normal surfaces. If a vertex v is such a projection, then let $s(v)$ be that surface such that for all normal surfaces σ projecting to v and integers $k > 0$, $k \cdot \sigma = s(v)$ if and only if $k = 1$. Then $s(v)$ is called a *vertex surface*, and represents an embedded surface in M.

It turns out that many vertex surfaces are topologically interesting, but some more interesting than others. In particular, boundary-parallel surfaces are only mildly interesting from our point of view. Therefore, Q-normal coordinates are more commensurate with our needs. Q-normal coordinates, moreover, only lie in $PR^{3\cdot t-1}$! In Q-normal coordinates, one only records the quads occurring in a normal surface, throwing away the triangle information. In [10] one finds that throwing away these triangle coordinates forces one to disregard vertex-linking surfaces, which of course we would like to do. This is the content of

Theorem 26 ([10], Thm. 1)**.** *Every normal surface yields an admissible solution to the Q-matching equations. Conversely, an admissible solution to the Q-matching equations is yielded by a* unique *normal surface with no vertex-linking ("trivial") components.*

That's enough to begin the proofs.

Proof of Lemma 24. Suppose M is a medium Seifert fibering with nonempty boundary, and suppose T finitely triangulates M.

M's base orbifold has at least one boundary component S. Suppose there were no essential simple arc from S to itself. Then M has a disc with no cone points for its base orbifold, and M is a solid torus. But solid tori are ∂-compressible, contrary to our assumptions on M. Hence there is an essential simple arc from S to itself.

The vertical fiber a over this arc is an annulus. a is essential, so it isotopes to a normal annulus. Let A be such an annulus such that the number of intersections of A with the 1-skeleton of T is minimal among normal surfaces isotopic to a. That is, let A have least weight in its isotopy class.

Now, every vertex surface in $\mathcal{C}(A)$ is an essential annulus or an essential torus (Cor. 6.8, [8]). There are no essential tori, by assumption. Consequently, each vertex surface in $\mathcal{C}(A)$ is an essential annulus. The proof of Theorem 2 in [10] shows that every two-sided vertex surface in $\mathcal{C}(A)$ is isotopic to a Q-vertex surface. Thus T admits some essential Q-vertex annulus. □

Proof of Lemma 25. Suppose M is a medium Seifert fibering with at least two boundary tori, and suppose T is a finite triangulation of M.

M's base orbifold now, by assumption, has at least two boundary components. Up to isotopy, there is a unique essential simple arc running between them. Let a be a vertical fiber over such an arc, a non-separating annulus. Since a is non-separating, it is essential. So it isotopes to a normal annulus. Let A be a least weight such annulus. Again, every vertex surface in $\mathcal{C}(A)$ is an essential annulus (or an essential torus, of which we've assumed there are none). Furthermore, every such essential annulus is isotopic to a Q-vertex surface, as above. So we just need a *non-separating* vertex annulus in $\mathcal{C}(A)$.

If there is a horizontal vertex annulus in $\mathcal{C}(A)$, then that is a non-separating vertex annulus in $\mathcal{C}(A)$.

Otherwise, A is a sum of some vertical vertex annuli in $\mathcal{C}(A)$.

The geometric sum (see [9], pp. 136–7 or [8], Fig. 2.1, p. 363) on the boundary just resolves intersections $\times \to \asymp$, which preserves homology mod 2. Therefore, for all normal surfaces S, S', $\partial(S+S') \equiv \partial(S) + \partial(S')$ in $H_1(\partial M; \mathbb{Z}/2\mathbb{Z})$.

The boundaries of separating annuli in M are 0 in $H_1(\partial M; \mathbb{Z}/2\mathbb{Z})$, but ∂A is not 0 in this homology. So A is not a sum of separating annuli. Consequently one of the summands must be non-separating, and be a non-separating vertex annulus in $\mathcal{C}(A)$. □

4 Further Directions

The algorithm presented here is convenient for demonstrating non-hyperbolicity, but it only works for cusped manifolds. Closed fault-testing is already implemented, but the absence of closed faults in a closed 3-manifold only implies that said 3-manifold is either hyperbolic or small Seifert-fibered. Tao Li and, independently, J. Hyam Rubinstein have given algorithms for locating a vertical torus in such a space, cutting along which torus yields a solid torus and a cusped small Seifert fibering, the detection of which are implemented. So the time is ripe to implement a rudimentary algorithm for hyperbolicity testing for arbitrary orientable compact 3-manifolds of Euler characteristic 0.

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