The number $e$ exists

1. The Rules
   Today we finish off our set of calculus rules by differentiating $b^x$ and $\log_b x$. Let us dive right in:

   \[
   (b^x)' = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h \to 0} \frac{b^x \cdot b^h - b^x}{h} = \lim_{h \to 0} b^x \cdot \frac{b^h - 1}{h} = b^x \lim_{h \to 0} \frac{b^h - 1}{h}
   \]

   Hmm. So if we let $l(b) = \lim_{h \to 0} \frac{b^h - 1}{h}$, then

   \[
   (b^x)' = b^x l(x).
   \]

   But what is $l(x)$? Well, one thing is for sure:

   (*) If there is a number $e$ such that $l(e) = 1$, then $(e^x)' = e^x$.

   Let us write $\ln x = \log_e x$ for such a number $e$. Then

   \[
   b^x = (e^{\ln b})^x = e^{(\ln b) \cdot x} = e^{x \ln b}.
   \]

   So

   \[
   (b^x)' = (e^{x \ln b})' = e^{x \ln b} \cdot (x \ln b)' = e^{x \ln b} \cdot \ln b = b^x \cdot \ln b.
   \]

   But $(b^x)' = b^x l(b)$. So it must be that $l(b) = \ln b$.

   What is more, since $\ln x$ and $\exp(x) = e^x$ are inverses of each other,

   \[
   \ln'(x) = \frac{1}{\exp'(\ln x)}.
   \]

   But $\exp'(x) = \exp(x)$ by assumption. So

   \[
   \ln'(x) = \frac{1}{\exp'(\ln x)} = \frac{1}{x}.
   \]

   Finally, the change-of-base formula for logarithms says that $\log_b x = \frac{\ln x}{\ln b}$; therefore,

   \[
   \log_b'x = \frac{1}{\ln b} \cdot \ln'(x) = \frac{1}{x} \ln b.
   \]

   In short,

   \[
   (e^x)' = e^x, \quad (\ln x)' = \frac{1}{x}, \quad (b^x)' = b^x \ln b, \text{ and } \quad (\log_b x)' = \frac{1}{x \ln b}.
   \]

   if there exists some number $e$ such that $l(e) = 1$. But we need to prove this.

2. Survey of Definitions of $e$
   To prove this, we could, of course, construct the number $e$ using any of the various definitions given in the literature. Some representative examples include

   [P] $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$, where the factorial $n!$ is given by $n! = 1 \cdot \cdots \cdot n$.

   [L] $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$.

   [A] $e$ is the number such that the area between the hyperbola $y = 1/x$ and the interval $[1, e]$ is 1. (put figure here)

   [C]

   \[
   e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \cdots}}}}}
   \]

   [E] $e = \exp(1)$, where $\exp$ is the unique function such that $\exp'(x) = \exp(x)$ and $\exp(0) = 1$. 

Unfortunately,

- showing \( l(e) = 1 \) is hard to prove using definition [P];
- it’s hard to show that the limit in [L] exists;
- it’s hard to show that the function \( \exp \) in definition [E] exists, and
- it’s not immediately clear what any of these definitions have to do with \( l(x) \).

For this reason, we will not construct \( e \). Instead, we will only convince ourselves that it exists. That is, all we will show is that there must be some number \( e \) such that \( l(e) = 1 \).

## 3. IVT and the Plan of Attack

We haven’t discussed any methods to show the existence of numbers, and it’s high time for us to do so. To that end, let us note the following intuitive fact:

If a line \( L \) separates two points \( P, Q \) into different halves of the plane, and if \( c \) is a continuous curve joining \( P \) to \( Q \), then \( c \) crosses \( L \) at some point. (See figure.)

We may use this fact to prove the following theorem, which has the fancy name

**Intermediate Value Theorem (IVT)**

If \( f \) is a continuous real function on an interval \([a, b]\), then \( f \) takes on all values between \( f(a) \) and \( f(b) \).

That is, if a number \( v \) lies between \( f(a) \) and \( f(b) \), there is \( c \in [a, b] \) such that \( f(c) = v \).

**Proof**

Since \( v \) is between \( f(a) \) and \( f(b) \), the points \((a, f(a))\) and \((b, f(b))\) lie on opposite sides of the line \( y = v \). (See figure.) Since \( f \) is continuous, its graph is a continuous curve. Therefore, the graph of \( f \) intersects the line \( y = v \) at some point \((c, v)\) for some \( c \in [a, b] \). But since \((c, v)\) is on the graph of \( f \), \((c, v) = (c, f(c)) \). Therefore, \( v = f(c) \) for some \( c \in [a, b] \).

We may use the IVT to show the existence of \( e \) as follows: if we can show that

1. the function \( l(a) \) is continuous in \( a \),
2. \( l(1) = 0 \), and
3. \( l(d) > 1 \) for some \( d \),

then on \([1, d] \) the function \( l \) must take on all values between 0 and \( l(d) \); in particular, it must take on the value 1 at some point \( e \) between 1 and \( d \).

So our plan of attack is to prove (1), (2), and (3). Then we will have shown that there is a number \( e \) such that \( l(e) = \lim_{h \to 0} \frac{e^h - 1}{h} = 1 \).

## 4. Proof of Existence

It is easy enough to prove (2):

\[
l(1) = \lim_{h \to 0} \frac{1^h - 1}{h} = \lim_{h \to 0} \frac{0}{h} = 0.
\]

To prove (1), we need to show

\[
\lim_{x \to a} l(x) = l(a).
\]

We may rewrite this condition as

\[
\lim_{x \to a} l(x) - l(a) = 0.
\]

Now, since we suspect that \( l(b) = \ln(b) \), we suspect that \( l(x) - l(a) = l(x/a) \). We can prove this straightaway:

**Lemma**

\( l(b) - l(a) = l(b/a) \) for all \( a, b \).

**Proof**

\[
l(b) - l(a) = \lim_{h \to 0} \frac{b^h - 1}{h} - \lim_{h \to 0} \frac{a^h - 1}{h}
\]

\[= \lim_{h \to 0} \frac{b^h - 1}{h} - \frac{a^h - 1}{h}
\]

\[= \lim_{h \to 0} \frac{b^h - 1 - (a^h - 1)}{h}
\]

\[= \lim_{h \to 0} \frac{b^h - 1 - a^h + 1}{h}
\]

\[= \lim_{h \to 0} \frac{b^h - a^h}{h}
\]

\[= \lim_{h \to 0} \frac{b^h}{h} - \frac{a^h}{h}
\]

\[= \lim_{h \to 0} \frac{b^h - 1}{a^h}
\]
\[
\lim_{h \to 0} \left( a^h \cdot \frac{\left( \frac{b}{a} \right)^h - 1}{h} \right) = \lim_{h \to 0} a^h \cdot \left( \lim_{h \to 0} \frac{\left( \frac{b}{a} \right)^h - 1}{h} \right)
= \lim_{h \to 0} a^h \cdot \left( \lim_{h \to 0} \frac{\left( \frac{b}{a} \right)^h - 1}{h} \right)
= 1 \cdot l(b/a) = l(b/a)
\]

QED

So to show that \( l \) is continuous, instead of showing \( \lim_{x \to a} l(x) - l(a) = 0 \), we may show that \( \lim_{x \to a} l(x/a) = 0 \).

But as \( x \to a \), \( x/a \to 1 \). So, again, to show continuity of \( l \), it will suffice to show that \( \lim l(q) = 0 \). That is, \( l \) is continuous precisely if the following holds:

**Lemma**

\[
\lim_{q \to 1} \lim_{h \to 0} \frac{q^h - 1}{h} = 0.
\]

**Proof**

We use the following trick: taking the limit as \( h \to 0^+ \) is equivalent to letting

\[
\frac{1}{n + 1} < h < \frac{1}{n},
\]

and taking the limit as \( n \to \infty \). (A similar trick applies when we take the limit as \( h \to 0^- \).)

Since \( \frac{1}{n + 1} < h < \frac{1}{n} \),
\[
q^{1/(n+1)} - 1 < q^h - 1 < q^{1/n} - 1.
\]

For convenience, let us write \( r = q^{1/(n+1)} \) and \( R = q^{1/n} \); then \( r^{n+1} = R^n = q \), and
\[
r - 1 < q^h - 1 < R - 1.
\]

We divide by \( r^{n+1} - 1 = q - 1 = R^n - 1 \);
so
\[
\frac{r - 1}{r^{n+1} - 1} < \frac{q^h - 1}{q - 1} < \frac{R - 1}{R^n - 1}.
\]

Now,
\[
x^m - 1 = x^{m-1} + x^{m-2} + \cdots + x + 1,
\]

since
\[
(x - 1)(x^{m-1} + x^{m-2} + \cdots + x + 1)
= x^m + x^{m-1} + \cdots + x
- x^{m-1} - \cdots - x - 1
= x^m - 1.
\]

So indeed \( \frac{x^m - 1}{x - 1} = x^{m-1} + \cdots + x + 1 \).

Therefore,
\[
\frac{1}{r^{n+1} - 1} < \frac{q^h - 1}{q - 1} < \frac{1}{R^{n-1} + \cdots + 1}.
\]

Assume that \( q > 1 \) (a similar argument works when assuming \( q < 1 \)). Then \( r = q^{1/(n+1)} > 1 \) and \( R = q^{1/n} > 1 \).

Therefore,
\[
r^n + \cdots + r^n = (n + 1)r^n.
\]

so
\[
\frac{1}{r^{n+1} - 1} < \frac{q^h - 1}{q - 1} < \frac{1}{R^{n-1} + \cdots + 1}.
\]

Similarly,
\[
R^{n-1} + \cdots + 1 > 1 + \cdots + 1 = n,
\]

so
\[
\frac{1}{R^{n-1} + \cdots + 1} < \frac{1}{n}.
\]

Consequently we get the simpler inequality
\[
\frac{1}{(n + 1)q^{n/(n+1)}} < \frac{q^h - 1}{q - 1} < \frac{1}{n}.
\]

Multiplying both sides now by \( q - 1 \), we get
\[
\frac{q - 1}{(n + 1)q^{n/(n+1)}} < q^h - 1 < \frac{q - 1}{n}.
\]

Now
\[
n < \frac{1}{h} < n + 1
\]

since \( \frac{1}{n + 1} < h < \frac{1}{n} \). Therefore,
\[
\frac{n(q - 1)}{(n + 1)q^{n/(n+1)}} < \frac{q^h - 1}{h} < \frac{(q - 1)(n + 1)}{n}.
\]

We now finally take the limit as \( n \to \infty \) and as \( q \to 1 \):
\[
\frac{q - 1}{q} \leq \lim_{n \to \infty} \frac{q^h - 1}{h} \leq q - 1,
\]

\[
0 \leq \lim_{q \to 1} \lim_{n \to \infty} \frac{q^h - 1}{h} \leq 0,
\]

so necessarily

\[
\lim_{q \to 1} \lim_{n \to \infty} \frac{q^h - 1}{h} = 0,
\]

which was what we wanted.

Therefore \( l \) is continuous. So (1) is true.

Lastly we need to show (3), that \( l(d) > 1 \) for some \( d \). First, let’s show another property of \( l \) we suspect is true:

**Lemma**

\( l(b) + l(a) = l(b \cdot a) \)

**Proof**

\[-l(a) = 0 - l(a) = l(1) - l(a) = l(1/a) \]

\[ l(b) - l(a) = l(b) - (l(a)) = l(b) - l(1/a) = l(b/(1/a)) = l(b \cdot a). \]

QED

Using this property we see that it suffices to find \( c \) such that \( l(c) > 0 \); then there must be some whole number \( N \) such that \( N \cdot l(c) > 1 \). But \( N \cdot l(c) = l(c^N) \); so just pick \( d = c^N \).

It will suffice to show, then, that

**Lemma**

\[ \lim_{b \to \infty} l(b) \geq 1 \]

(for then \( l(c) \geq 1 > 0 \) for all \( c \) sufficiently large).

**Proof**

Since \( l \) is continuous, it will suffice to show that

\[
\lim_{a \to \infty} \lim_{n \to \infty} \frac{a^{1/n} - 1}{1/n} \geq 1.
\]

Let \( \rho = a^{1/n} \). Then

\[
n(a^{1/n} - 1) = \frac{n}{a - 1} \frac{(a^{1/n} - 1)(a - 1)}{\rho^{n-1} + \ldots + 1} = \frac{n}{\rho^{n-1}} = \frac{a - 1}{\rho^{n-1}} = \frac{a - 1}{a^{(n-1)/n}}.
\]

So

\[
\lim_{a \to \infty} \lim_{n \to \infty} \frac{a^{1/n} - 1}{1/n} \geq \lim_{a \to \infty} \lim_{n \to \infty} \frac{a - 1}{a^{(n-1)/n}} = \lim_{a \to \infty} \frac{a - 1}{a} = 1.
\]

QED

Consequently, (1), (2), and (3) are true. Hence the number \( e \) exists by the IVT.