Figure 3.1 GDP and Capital per Worker, 2009

Source: Calculations based on Heston et al. (2010).
Production Function

\[ Y = F(K, L) \]

Constant Returns to Scale:
\[ F(zK, zL) = zF(K, L) \]

\[ \frac{1}{L}Y = \frac{1}{L}F(K, L) = F\left(\frac{K}{L}, 1\right) \]

\[ y = \frac{Y}{L} = F(k, 1) = f(k) \]

\[ MPK = f(k + 1) - f(k) \]

\[ MPK = \frac{df(k)}{dk} \]
Figure 3.2 A Production Function with Diminishing Marginal Product of Capital
Cobb-Douglas Production Function

\[ Y = F(K, L) = AK^\alpha L^{1-\alpha} \]

\[ \frac{1}{L} Y = \frac{1}{L} F(K, L) = F\left(\frac{K}{L}, 1\right) = A\left(\frac{K}{L}\right)^\alpha (1)^{1-\alpha} \]

\[ y = Ak^\alpha \]
Capital’s Share in Income

Capital's Share in Income = \( \frac{MPK \ast K}{Y} \)

For Cobb-Douglas, \( \alpha \) is this share:

\( MPK = \alpha AK^{\alpha-1} L^{1-\alpha} \)

So, Capital's Share = \( \frac{MPK \ast K}{Y} = \frac{\alpha AK^{\alpha-1} L^{1-\alpha} K}{Y} \)

= \( \frac{\alpha AK^{\alpha} L^{1-\alpha}}{Y} = \frac{\alpha Y}{Y} = \alpha \)
Figure 3.3 Capital’s Share of Income in a Cross-Section of Countries

Source: Bernanke and Gürkaynak (2002), table 10 and note 18.
Table 3.1 Agricultural Land as a Fraction of Total Wealth in the United Kingdom

<table>
<thead>
<tr>
<th>Year</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>1688</td>
<td>64%</td>
</tr>
<tr>
<td>1798</td>
<td>55%</td>
</tr>
<tr>
<td>1885</td>
<td>18%</td>
</tr>
<tr>
<td>1927</td>
<td>4%</td>
</tr>
<tr>
<td>1958</td>
<td>3%</td>
</tr>
</tbody>
</table>
The Solow Growth Model

\[ \Delta K = I - D \]
\[ \Delta k = i - d \]
\[ i = \gamma y \text{ (investment)} \]
\[ d = \delta k \text{ (depreciation)} \]

Evolution of k over time:
\[ \Delta k = \gamma y - \delta k \]
\[ \Delta k = \gamma f(k) - \delta k \]
Digression on Measuring Change Over Time

Discrete time: \[ \Delta x_t = x_{t+1} - x_t \]

Growth Rate: \[ \hat{x} = \frac{\Delta x_t}{x_t} = \frac{x_{t+1} - x_t}{x_t} \]

Continuous time: \[ \dot{x}_t = \frac{dx}{dt} \]

Growth Rate: \[ \hat{x} = \frac{\dot{x}}{x_t} \]
Figure 3.4 The Steady State of the Solow Model
Figure 3.5 Determination of Steady-State Weight
Figure 3.6 Effect of Increasing the Investment Rate on the Steady State

Note: $\gamma_2 > \gamma_1$
Steady State Solution for Cobb Douglas Production Function

\[ \Delta k = \gamma A k^\alpha - \delta k \]
\[ \Delta k = 0 \text{ implies} \]
\[ \gamma A \left( k^{ss} \right)^\alpha = \delta k^{ss} \]

\[ k^{ss} = \left( \frac{\gamma A}{\delta} \right)^{1/1-\alpha} \]

Plug in to production function to obtain:

\[ y^{ss} = A \left( k^{ss} \right)^\alpha = A^{1/1-\alpha} \left( \frac{\gamma}{\delta} \right)^{\alpha/1-\alpha} \]
Steady State Solution for Cobb Douglas Production Function

\[ y^{ss} = A^{1/1-\alpha} \left( \frac{\gamma}{\delta} \right)^{\alpha/1-\alpha} \]

As the investment share (\( \gamma \)) rises, steady-state output per worker (\( y^{ss} \)) increases; As the depreciation rate (\( \delta \)) rises, steady-state output per worker falls.

Can we use this to explain income differences across countries?
Explaining Income Differences

Assume steady state:

\[ y_{is}^{ss} = A^{1/1-\alpha} \left( \frac{\gamma_i}{\delta} \right)^{\alpha/1-\alpha} ; \quad y_{js}^{ss} = A^{1/1-\alpha} \left( \frac{\gamma_j}{\delta} \right)^{\alpha/1-\alpha} \]

\[ \frac{y_{is}^{ss}}{y_{js}^{ss}} = \frac{A^{1/1-\alpha} \left( \frac{\gamma_i}{\delta} \right)^{\alpha/1-\alpha}}{A^{1/1-\alpha} \left( \frac{\gamma_j}{\delta} \right)^{\alpha/1-\alpha}} = \left( \frac{\gamma_i}{\gamma_j} \right)^{\alpha/1-\alpha} \]

If \( \alpha = 1/3 \), \( \gamma_i = 0.20 \), and \( \gamma_j = 0.05 \) we obtain:

\[ \frac{y_{is}^{ss}}{y_{js}^{ss}} = \left( \frac{0.2}{0.05} \right)^{1/2} = 4^{1/2} = 2 \]
Figure 3.7 Predicted versus Actual GDP per Worker

Source: Author’s calculations using data from Heston, Summers, and Aten (2011).
Speed of Convergence to the Steady State

\[ \Delta k = \gamma A k^\alpha - \delta k \]

\[ \hat{k} = \frac{\Delta k}{k} = \gamma A k^{\alpha - 1} - \delta \]

Shows speed with which the economy approaches the steady state.
Figure 3.10 Speed of Convergence to the Steady State

\[ \gamma A k^{\alpha - 1} \]

\[ k^* \]

\[ \delta \]

Capital per worker (k)
Figure 3.8 Saving Rate by Decile of Income per Capita

Average saving rate, 2009

Decile of GDP per capita, 2009
**Figure 3.9** Solow Model with Saving Dependent on Income Level