Problem Set 3 Solutions

4. Hotelling’s Lemma

(a) The Lagrangian for the firm’s problem is

$$L(y, k, l, \lambda) = py - rk - wk + \lambda[f(k, l) - y].$$

(b) According to the Kuhn-Tucker theorem, the values $y^*, k^*, l^*$ that solve the firm’s problem, together with the associated value $\lambda^*$ for the multiplier, must satisfy the first-order conditions

$$L_1(y^*, k^*, l^*, \lambda^*) = p - \lambda^* = 0,$$

$$L_2(y^*, k^*, l^*, \lambda^*) = -r + \lambda^* f_1(k^*, l^*) = 0,$$

and

$$L_3(y^*, k^*, l^*, \lambda^*) = -w + \lambda^* f_2(k^*, l^*) = 0,$$

the constraint

$$L_4(y^*, k^*, l^*, \lambda^*) = f(k^*, l^*) - y^* \geq 0,$$

the nonnegativity condition

$$\lambda^* \geq 0,$$

and the complementary slackness condition

$$\lambda^*[f(k^*, l^*) - y^*] = 0.$$

(c) Assume that the output and input prices $p$, $r$, and $w$ and the production function $f$ are such that it is possible to solve uniquely for the values of $y^*$, $k^*$, $l^*$, and $\lambda^*$ in terms of the parameters $p$, $r$, and $w$. Then the profit function, defined as

$$\pi(p, r, w) = \max_{y,k,l} py - rk - wl \text{ subject to } f(k, l) \geq y,$$

can be evaluated as

$$\pi(p, r, w) = py^*(p, r, w) - rk^*(p, r, w) - wl^*(p, r, w).$$
or, using the complementary slackness condition,

\[
\pi(p, r, w) = p y^*(p, r, w) - r k^*(p, r, w) - w l^*(p, r, w) + \lambda^*(p, r, w) \{ f[k^*(p, r, w), l^*(p, r, w)] - y^*(p, r, w) \},
\]

where \(y^*(p, r, w)\) is the firm’s supply function, \(k^*(p, r, w)\) and \(l^*(p, r, w)\) are the factor demand curves, and \(\lambda^*(p, r, w)\) is the function describing the associated values of the Lagrange multiplier. The envelope theorem says that in differentiating this expression for the profit function through by each argument, one can ignore the dependence of \(y^*, k^*, l^*, \) and \(\lambda^*\) on those parameters, and simply write

\[
\pi_1(p, r, w) = y^*(p, r, w),
\]

\[
\pi_2(p, r, w) = -k^*(p, r, w),
\]

and

\[
\pi_3(p, r, w) = -l^*(p, r, w),
\]

which are the same results described by Hotelling’s lemma.

5. Shephard’s Lemma

(a) The Lagrangian for this problem is

\[
L(k, l, \mu) = r k + w l - \mu \{ f(k, l) - y \}.
\]

(b) According to the Kuhn-Tucker theorem, the values \(k^*\) and \(l^*\) that solve the firm’s problem, together with the associated value \(\mu^*\) for the multiplier, must satisfy the first-order conditions

\[
L_1(k^*, l^*, \mu^*) = r - \mu^* f_1(k^*, l^*) = 0
\]

and

\[
L_2(k^*, l^*, \mu^*) = w - \mu^* f_2(k^*, l^*) = 0,
\]

the constraint

\[
L_3(k^*, l^*, \mu^*) = f(k^*, l^*) - y \geq 0,
\]

the nonnegativity condition

\[
\mu^* \geq 0,
\]

and the complementary slackness condition

\[
\mu^* \{ f(k^*, l^*) - y \} = 0.
\]

(c) Assume that the output requirement \(y\), the input prices \(r\) and \(w\), and the production function \(f\) are such that it is possible to solve uniquely for the values of \(k^*, l^*,\) and \(\mu^*\) in terms of the parameters \(r, w,\) and \(y\). Then the cost function,
defined as
\[ c(r, w, y) = \min_{k,l} rk + wl \text{ subject to } f(k, l) \geq y, \]
can be evaluated as
\[ c(r, w, y) = rk^*(r, w, y) + wl^*(r, w, y) \]
or, using the complementary slackness condition,
\[ c(r, w, y) = rk^*(r, w, y) + wl^*(r, w, y) - \mu^*(r, w, y)\{f[k^*(r, w, y), l^*(r, w, y)] - y\}, \]
where \( k^*(r, w, y) \) and \( l^*(r, w, y) \) now represent the conditional factor demand curves and \( \mu^*(r, w, y) \) is the function describing the associated values of the Lagrange multiplier. The envelope theorem says that in differentiating this expression for the cost function through by each argument, one can ignore the dependence of \( k^*, l^* \), and \( \mu^* \) on those parameters, and simply write
\[ c_1(r, w, y) = k^*(r, w, y) \]
and
\[ c_2(r, w, y) = l^*(r, w, y), \]
which are the same results described by Shephard’s lemma.

6. Roy’s Identity

(a) The Lagrangian for this problem is
\[ L(c_1, c_2, \lambda) = U(c_1, c_2) + \lambda(I - p_1c_1 - p_2c_2). \]

(b) According to the Kuhn-Tucker theorem implies that, the values \( c_1^* \) and \( c_2^* \) that solve the consumer’s problem, together with the associated value \( \lambda^* \) for the multiplier, must satisfy the first-order conditions
\[ L_1(c_1^*, c_2^*, \lambda^*) = U_1(c_1^*, c_2^*) - \lambda^* p_1 = 0 \]
and
\[ L_2(c_1^*, c_2^*, \lambda^*) = U_2(c_1^*, c_2^*) - \lambda^* p_2 = 0, \]
the constraint
\[ L_3(c_1^*, c_2^*, \lambda^*) = I - p_1c_1^* - p_2c_2^* \geq 0, \]
the nonnegativity condition
\[ \lambda^* \geq 0, \]
and the complementary slackness condition
\[ \lambda^*(I - p_1c_1^* - p_2c_2^*) = 0. \]

(c) Assume that income \( I \), and goods prices \( p_1 \) and \( p_2 \), and the utility function \( U \) are
such that it is possible to solve uniquely for the values of $c_1^*$, $c_2^*$, and $\lambda^*$ in terms of the parameters $I$, $p_1$, and $p_2$. Then the indirect utility function, defined as

$$v(p_1, p_2, I) = \max_{c_1, c_2} U(c_1, c_2) \text{ subject to } I \geq p_1 c_1 + p_2 c_2,$$

can be evaluated as

$$v(p_1, p_2, I) = U[c_1^*(p_1, p_2, I), c_2^*(p_1, p_2, I)]$$

or, using the complementary slackness condition,

$$v(p_1, p_2, I) = U[c_1^*(p_1, p_2, I), c_2^*(p_1, p_2, I)] + \lambda^*(p_1, p_2, I)[I - p_1 c_1^*(p_1, p_2, I) - p_2 c_2^*(p_1, p_2, I)],$$

where $c_1^*(p_1, p_2, I)$ and $c_2^*(p_1, p_2, I)$ define the Marshallian demand curves for the two goods and $\lambda^*(p_1, p_2, I)$ describes the associated values for the Lagrange multiplier. The envelope theorem says that in differentiating this last expression for the indirect utility function through by each argument, one can ignore the dependence of $c_1^*$, $c_2^*$, and $\lambda^*$ on those parameters, and simply write

$$v_1(p_1, p_2, I) = -\lambda^*(p_1, p_2, I)c_1^*(p_1, p_2, I),$$

$$v_2(p_1, p_2, I) = -\lambda^*(p_1, p_2, I)c_2^*(p_1, p_2, I),$$

and

$$v_3(p_1, p_2, I) = \lambda^*(p_1, p_2, I).$$

Dividing the first and second of these equations by the third leads directly to a statement of Roy’s identity.

7. The Slutsky Equation

(a) The Lagrangian for this problem is

$$L(c_1, c_2, \mu) = p_1 c_1 + p_2 c_2 + \mu[U(c_1, c_2) - \bar{U}].$$

(b) According to the Kuhn-Tucker theorem, the values $c_1^*$ and $c_2^*$ that solve this problem, together with the associated value $\mu^*$ for the multiplier, must satisfy the first-order conditions

$$L_1(c_1^*, c_2^*, \mu^*) = p_1 - \mu^* U_1(c_1^*, c_2^*) = 0$$

and

$$L_2(c_1^*, c_2^*, \mu^*) = p_2 - \mu^* U_2(c_1^*, c_2^*) = 0,$$

the constraint

$$L_3(c_1^*, c_2^*, \mu^*) = U(c_1^*, c_2^*) - \bar{U} \geq 0,$$

the nonnegativity condition

$$\mu^* \geq 0,$$
and the complementary slackness condition
\[ \mu^*[U(c_1^*, c_2^*) - \bar{U}] = 0. \]

(c) Assume that income $I$, goods prices $p_1$ and $p_2$, the utility function $U$, and the utility level $\bar{U}$ are such that it is possible to solve uniquely for the values of $c_1^*$, $c_2^*$, and $\mu^*$ in terms of the parameters $\bar{U}$, $p_1$, and $p_2$. Now the functions $c_1^* = h_1^*(p_1, p_2, \bar{U})$ and $c_2^* = h_2^*(p_1, p_2, \bar{U})$ describing the optimal choices are the Hicksian demand curves for the two goods. Along with these functions, define the expenditure function

\[ e(p_1, p_2, \bar{U}) = \min_{c_1, c_2} p_1 c_1 + p_2 c_2 \text{ subject to } U(c_1, c_2) \geq \bar{U}. \]

Under most circumstances, it will be the case that the Marshallian and Hicksian demand curves coincide at the point where $I = e(p_1, p_2, \bar{U})$, so that the income $I$ from the utility maximization problem equals the expenditure required to attain the utility level $v(p_1, p_2, I) = \bar{U}$. This result can be summarized by stating that

\[ h_i^*(p_1, p_2, \bar{U}) = c_i^*(p_1, p_2, e(p_1, p_2, \bar{U})) \]

for all values of $p_1$, $p_2$, and $\bar{U}$ and each $i = 1, 2$. Differentiating both sides of this expression by $p_i$ and using the chain rule on the right-hand side, yields

\[ \frac{\partial h_i^*(p_1, p_2, \bar{U})}{\partial p_i} = \frac{\partial c_i^*(p_1, p_2, e(p_1, p_2, \bar{U}))}{\partial p_i} + \frac{\partial c_i^*(p_1, p_2, e(p_1, p_2, \bar{U}))}{\partial I} \frac{\partial e(p_1, p_2, \bar{U})}{\partial p_i}. \]

However, the envelope theorem, when applied to evaluate the derivatives of the expenditure function, implies that

\[ \frac{\partial e(p_1, p_2, \bar{U})}{\partial p_i} = h_i^*(p_1, p_2, \bar{U}) = c_i^*(p_1, p_2, e(p_1, p_2, \bar{U})). \]

Substituting this last expression, together with $I = e(p_1, p_2, \bar{U})$ and $v(p_1, p_2, I) = \bar{U}$, into the one above it and rearranging yields the Slutsky equation

\[ \frac{\partial c_i^*(p_1, p_2, I)}{\partial p_i} = \frac{\partial h_i^*(p_1, p_2, v(p_1, p_2, I))}{\partial p_i} - \frac{\partial c_i^*(p_1, p_2, I)}{\partial I} \frac{\partial e(p_1, p_2, I)}{\partial p_i} \]

for each $i = 1, 2$. 

5
1) a) Let \( x^* \) solve Problem 1. Then, \( \forall x \in C \equiv \{ \mathbf{x} : g_j(x) \leq 0 \} \)
\[ f(x^*) \geq f(x) \] Since \( T \) is non-decreasing,
\[ T(f(x^*)) \geq T(f(x)) \]
and \( x^* \) solves Problem 2.

b) Let \( x^* \) solve Problem 2. Then, \( \forall x \in C \)
\[ T(f(x^*)) \geq T(f(x)) \] Since \( T \) is strictly increasing \( \alpha \geq \beta \) \( \Leftrightarrow \) \( T(\alpha) \geq T(\beta) \)
and \( T(\alpha) = T(\beta) \) \( \Rightarrow \) \( \alpha = \beta \). Thus, \( T(f(x^*)) \geq T(f(x)) \)
\( \Rightarrow \) \( f(x^*) \geq f(x) \) and \( T(f(x^*)) = T(f(x)) \)
so \( f(x^*) \geq f(x) \). Thus \( x^* \) solves Problem 1.

c) Now, \( g_j \) being quasiconvex \( \Rightarrow \) \( C \) is convex. Since
\( T(f) \) is strictly concave \( \forall x \in C \) and \( C \) is non-empty, \( \exists \) a
unique \( x^* \in C \) such that \( T(f(x^*)) = T(f(x)) \) \( \forall x \in C \).

From part b) we know that \( x^* \) must solve Problem 1. Thus, \( x^* \).

Then \( f(x) \geq f(x^*) \) \( \forall x \in C \)
\( \Rightarrow \) \( T(f(x)) \geq T(f(x^*)) \)
\( \forall x \in C \). Since \( T \) is non-decreasing. Thus, \( x^* \) is a

maximum solution to Problem 2. But then with \( x^* \)

a contradiction since the uniqueness theorem. Thus, there

Suppose \( \exists \) is a solution to Problem 1. From part a) it must be a solution to Problem 2. But the uniqueness
sense for that \( \exists x = x^* \). Thus, Problem 1 has a unique maximum.
d) Let $\hat{x}_1, \hat{x}_2$ be distinct maximizers of Problem 1.

Then from part a) $\hat{x}_1, \hat{x}_2$ solve Problem 2. Then convexity of the maximizer set implies $\hat{x} = \theta \hat{x}_1 + (1-\theta) \hat{x}_2$ solves Problem 2 for all $\theta \in [0,1]$. But then part b) states implies that $\hat{x}$ solves Problem 1 and so the set of maximizers of Problem 1 is convex.

e) Let $\hat{x}_1, \hat{x}_2$ be maximizers of Problem 2. Then from part a) consider $\hat{x} = \theta \hat{x}_1 + (1-\theta) \hat{x}_2$ for some $\theta \in [0,1]$.

$\hat{x}$ T(\theta) is quasiconcave, and so $T(f(x)) \geq \min \{T(f(x_1), T(x_2))\}$

and so $T$ Thus, $\hat{x}$ is a maximizer of Problem 2, and so the maximizer set is convex.

f) Suppose $\hat{x}_1, \hat{x}_2$ are distinct solutions to Problem 1, which is possible since $f$ is weakly concave but not necessarily strictly concave. Then part a) implies that $\hat{x}_1, \hat{x}_2$ are maximizers of Problem 2. Basically, T(\theta) is not necessarily strictly concave even though T is strictly increasing.

g) Let $x^*$ solve Problem 1 and suppose we have some point $\hat{x} \in C, \hat{x} \neq x^*$, where the value of $f$ at $\hat{x}$ is denoted by $y = f(x^*) \leq f(x^*)$. Suppose $T(y) = T(y) + y > y$.

Then $T(f(x^*)) = T(f(x^*))$ and hence from part a) we know $x^*$ solves Problem 2, so $x^*$ solves Problem 2 as well.
We know $x^*$ is the unique solution to Problem 1. Also suppose $x$ is some solution to Problem 2. Then part b) tells us that $x$ solves Problem 1 and so $x = x^*$. Thus Problem 2 has at most one solution. But from part a) we know that $x^*$ solves Problem 2. Thus, $x^*$ is the unique solution of Problem 2.

Suppose $x^* \in C$ solves Problem 1 and $Dg_j(x^*)$ has rank 1 for $j \in J$ where $B \subseteq J$ are the binding constraints. Then KKT tells us that

\[
\nabla f(x^*) = \sum_{j \in J} \lambda_j Dg_j(x^*) , \quad \lambda_j \geq 0 , \quad j \in J ,
\]

\[
\lambda_j g_j(x^*) = 0 .
\]

Now, notice that $D^T T(f(x)) = T'(f(x)) D f(x)$ and ro fix $y = r(f(x))$

\[
D T(f(x)) = T'(f(x)) D f(x^*) .
\]

Then, $D T(f(x)) = \lambda D f(x^*) = \sum_{j \in J} \lambda_j Dg_j(x^*)$.

Define $\tilde{\lambda}_j = \gamma \lambda_j$. Then, $D T(f(x)) = \sum_{j \in J} \tilde{\lambda}_j Dg_j(x^*)$,

\[
\tilde{\lambda}_j \geq 0 , \quad j \in J
\]

\[
\tilde{\lambda}_j = 0 , \quad j \notin B \quad \text{and} \quad \lambda_j g_j(x^*) = 0 .
\]

Thus, $x^*$ solves Problem 2.
i) Now, if \( x^* \) solves Problem 2, it satisfies \( \text{NOCQ} \) for Problem 2, and after
\[
\nabla T(f(x)) = \sum_{j \in J} \bar{\lambda}_j g_j(x^*) \quad \bar{\lambda}_j \geq 0 \quad \bar{\lambda}_j g_j(x^*) = 0
\]

But
\[
\nabla T(f(x)) = T(H(x)) \nabla f(x) = y \nabla f(x)
\]

when \( y = T'(f(x^*)) \cdot T^T > 0 \)

Thus,
\[
\nabla f(x) = \frac{1}{y} \nabla T(H(x)) = \sum_{j \in J} \frac{\bar{\lambda}_j g_j(x^*)}{y}
\]

\[
= \sum_{j \in J} \frac{\bar{\lambda}_j g_j(x^*)}{y}.
\]

Also, \( \bar{\lambda}_j \geq 0 \) since \( y > 0 \) and \( \lambda_j \geq 0 \). Finally, \( \bar{\lambda}_j g_j(x^*) = 0 \).

Thus, \( x^* \) solves Problem 1.

k) From parts i) and j) we know that the set of solutions to the KKT conditions are equivalent, and so the sufficient KKT conditions for Problem 1 extends to Problem 2.

l) Since \( T(f) \) is concave, both KKT conditions are sufficient. Parts i), k) and j) imply equivalence of the set of solutions to the KKT conditions \( f^* \) of each problem, and so the sufficient result also extends.
2) \( \max f(x_1, x_2) - w_1 x_1 - w_2 x_2 \)

\( \text{a)} \quad \nabla f^T: \begin{align*}
  f_1 - w_1 & \leq 0 \quad x_1 \geq 0 \quad x_1(f_1 - w_1) = 0 \\
  f_2 - w_2 & \leq 0 \quad x_2 \geq 0 \quad x_2(f_2 - w_2) = 0
\end{align*} \)

These are sufficient since \( D_x^2 f \) is neg. dy.

\( \text{b)} \quad \text{Since } f \text{ is } C^2, \text{ and } D_x^2 f \text{ is neg. dy. (since having non-zero determinant)}, \text{ we can use the Implicit Function Theorem:} \quad D_w x^* = - [D_x G]^T D_w G, \quad \text{where} \quad G = \begin{bmatrix}
  f_1 - w_1 \\
  f_1 - w_2
\end{bmatrix} \\
\)

\( \quad \text{Thus,} \quad D_w x^* = - [D_x f]^T \begin{bmatrix} 1 & 0 \\
  0 & -1
\end{bmatrix} \quad D_x^2 f = \begin{bmatrix} f_{11} f_{12} \\
  f_{12} f_{22}
\end{bmatrix} \)

\( \quad = \frac{1}{|D_x f|} \begin{bmatrix}
  f_{11} & -f_{12} \\
  -f_{12} & f_{22}
\end{bmatrix} \quad \text{and so} \quad \frac{\partial x^*_1}{\partial w_1} = \frac{f_{12}}{|D_x f|} < 0 \quad \text{Since } D_x f \text{ is neg. dy.} \)

\( \quad \text{c)} \quad \max f(x_1, x_2) - w_1 x_1 - w_2 x_2 \quad \text{st. } x_2 = b \quad b > 0 \)

\( \text{Poc:} \quad \begin{align*}
  f_1 - w_1 & = 0 \quad f_1(x^*_1, b) - w_1 x_1 = 0 \\
  f_2 - w_2 & = 0 \quad x_2 = b \quad f_{11} \geq 0 \quad f_{22} \leq 0 \quad \text{so} \quad \frac{\partial x^*_1}{\partial w} = - \frac{(-1)}{f_{11}(x^*_1, b) = f_{11}(x^*_1, b)} < 0
\end{align*} \)
Notice that \( \frac{1}{f_{11}} \leq 1 \) and 
\[ \frac{f_{12}}{(f_{11} f_{22} - f_{12}^2)} \geq 0. \]

Thus \( \frac{\partial x_i^*}{\partial w_1} \leq \frac{\partial x_i^*}{\partial w_1} \) for \( i = 1, \ldots, n \).

3) \( \max \mathbb{E} u = \alpha_1 \ln x_1 + \beta_1 \ln y_1 + \alpha_2 \ln x_2 + \beta_2 \ln y_2 \)

s.t. \( p_1 x_1 + q_1 y_1 \leq I_1 \)
\( p_2 x_2 + q_2 y_2 \leq I_2 \)

Notice that F-standard conditions are satisfied so will not have corner solutions.

\( L = U - \lambda_1 (p_1 x_1 + q_1 y_1 - I_1) - \lambda_2 (p_2 x_2 + q_2 y_2 - I_2) \)

\( \lambda_1 = \frac{\alpha_1}{\frac{p_1 x_1}{x_1}} = \frac{\beta_1}{y_1} > 0 \)
\( \lambda_2 = \frac{\alpha_2}{\frac{p_2 x_2}{x_2}} = \frac{\beta_2}{y_2} > 0 \)

\( \therefore \lambda_1 > \frac{p_1 x_1}{x_1} + \beta_1 q_1 y_1 = I_1 \Rightarrow \lambda_1 = \frac{p_1 x_1}{x_1} \quad \lambda_2 = \frac{p_2 x_2}{x_2} \)

\( x_1 = \frac{x_1}{\alpha_1 + \beta_1} \quad x_2 = \frac{x_2}{\alpha_1 + \beta_1} \)
\( y_1 = \frac{\beta_1}{\alpha_1 + \beta_1} \quad y_2 = \frac{\beta_2}{\alpha_1 + \beta_1} \)

These must be solutions since \( U \) is concave (sum of concave functions is concave) and the budget sets are convex.
b) \[ \frac{\partial U^*}{\partial I_1} = \lambda_1 = \frac{I_1}{\lambda_1} \frac{\partial \lambda_1}{\partial I_1} \quad \frac{\partial U^*}{\partial I_2} = \lambda_2 = \frac{I_2}{\lambda_2} \frac{\partial \lambda_2}{\partial I_2} = \frac{x_1 + x_2}{I_2} \]

Thus, if holding utility constant, \( \frac{dI_2}{dI_1} = -\frac{\partial U^*/\partial I_2}{\partial U^*/\partial I_1} \)

\[ = -\frac{a_1}{a_2} \]

\[ = -\frac{(x_1 + x_2)}{(x_1 + x_2 + \phi)} \cdot \frac{I_2}{I_1} \]

Thus, if the net rate of return is \( r \),

i.e. \( Y_2 = (1+r)Y_1 \), where \( Y_1 \) is the level in a savings account, say, then \( dY_2 = (1+r)dy_1 \), and

\[ (1+r) = \left( \frac{x_2 + \phi_1}{x_2 + \phi_2} \right) \frac{I_2}{I_1} \]

c) Clearly \( E \leq I_1, I_2 \) will make consumer worse off and \( E \geq I_1, I_2 \) will make him better off. \( \frac{dU^*}{dI_1} \bigg|_{E} = \min \{ x_1, \phi \} \)

\[ U^*(I_1, I_2) = \frac{\partial U^*}{\partial I_1} \bigg|_{E} \]

\[ \Rightarrow I_1, I_2 = E \]

\[ x_1 + x_2 = E \]

\[ x_2 = \phi \]

\[ \frac{dI_2}{dI_1} = \left( \frac{x_1 + x_2}{x_1 + x_2 + \phi} \right) \frac{I_2}{I_1} \]

\[ x_1 = x_2 + \phi \]

\[ \phi = \phi \]

\[ \frac{dI_2}{dI_1} \bigg|_{E} = \frac{-x_1}{x_2} \cdot \frac{I_2}{I_1} \]

Change in prices do not affect the regions since prices have been homothetic.