Taut Foliations, Positive 3-Braids, and the L-Space Conjecture

Siddhi Krishna

Boston College

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The L-Space Conjecture:

Suppose $Y$ is a closed, irreducible 3-manifold. The following are equivalent:
The L-Space Conjecture:

Suppose $Y$ is a closed, irreducible 3-manifold. The following are equivalent:

- $Y$ admits a taut foliation
- $Y$ is a non-L-space
- $\pi_1(Y)$ is not left orderable
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“geometry” “Heegaard-Floer homology” “algebra”
The L-Space Conjecture:
Suppose \( Y \) is a closed, irreducible 3-manifold. The following are equivalent:

- \( Y \) admits a taut foliation \( \text{“geometry”} \)
- \( Y \) is a non-L-space \( \text{“Heegaard-Floer homology”} \)
- \( \pi_1(Y) \) is not left orderable \( \text{“algebra”} \)
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Taut Foliations

Definition

A taut foliation is

- a decomposition of a manifold into codim-1 submanifolds, called leaves, such that
- there exists a simple closed curve meeting each leaf transversely
An Example

Fibered 3-Manifolds

Start with

- a compact, connected, oriented surface $F$
- $\varphi : F \to F$, a diffeomorphism of $F$
An Example

Fibered 3-Manifolds

Start with
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- $\varphi : F \rightarrow F$, a diffeomorphism of $F$

This determines:

$$M_\varphi = F \times I / \varphi = F \times I / ((x, 0) \sim (\varphi(x), 1))$$
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Fibered 3-Manifolds

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This determines:

$$M_\varphi = \frac{F \times I}{\varphi} = \frac{F \times I}{((x, 0) \sim (\varphi(x), 1))}$$

![Diagram showing fibered 3-manifold](image)
An Example

Fibered 3-Manifolds

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- a compact, connected, orientable surface $F$
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This determines:

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Fibered 3-Manifolds

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- $\varphi : F \to F$, a diffeomorphism of $F$

This determines:

$$M_{\varphi} = F \times I / \varphi = F \times I / ((x, 0) \sim (\varphi(x), 1))$$
Taut Foliations in the Literature

**Theorem (Thurston ’86):**
Compact leaves of taut foliations minimize genus in their homology class.
Taut Foliations in the Literature

Theorem (Thurston ’86): Compact leaves of taut foliations minimize genus in their homology class.

Theorem (Gabai ’87): The “Property R” Conjecture is true, i.e.

\[ K \subset S^3, \text{ and } S_0^3(K) \approx S^1 \times S^2, \text{ then } K \approx U. \]
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L-Spaces

“L-spaces are simple from the perspective of Heegaard Floer homology”
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**Definition**

A closed, irreducible 3-manifold $Y$ is an **L-space** if

$$\text{rk}(\widehat{HF}(Y; \mathbb{Z}/2\mathbb{Z})) = |H_1(Y; \mathbb{Z})| < \infty$$
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**Remark:** $\text{rk}(\widehat{HF}(Y; \mathbb{Z}/2\mathbb{Z})) \geq |H_1(Y; \mathbb{Z})|$ holds for $\mathbb{Q}HS^3$
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Examples: Lens Spaces $\subset$ Manifolds with Elliptic Geometry
The L-Space Conjecture Revisited

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The manifolds are "extra."
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These manifolds are “extra”.

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Evidence for the LSC

Theorem (Ozsváth and Szabó):
If \( Y \) admits a taut foliation, then \( Y \) is a non-L-space.
Evidence for the LSC

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Theorem: The L-Space Conjecture is true for graph manifolds.
Evidence for the LSC

**Theorem (Ozsváth and Szabó):**
If \( Y \) admits a taut foliation, then \( Y \) is a non-L-space.

**Theorem:** The L-Space Conjecture is true for **graph manifolds**.
**Proof:** Input by many!

For $Y$ is a closed, irreducible 3-manifold:

- $Y$ admits a taut foliation
- $Y$ is a non-L-space

The Ozsváth-Szabó method connects these two properties.
For $Y$ is a closed, irreducible 3-manifold:

$Y$ admits a taut foliation $\Rightarrow$ $Y$ is a non-L-space

Ozsváth-Szabó

Siddhi Krishna

Constructing Taut Foliations in Positive 3-Braid Exteriors
Focusing on Foliations

For $Y$ is a closed, irreducible 3-manifold:

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Ozsváth-Szabó

Questions: How do we
(1) identify non-L-spaces?
For $Y$ is a closed, irreducible 3-manifold: $Y$ admits a taut foliation $Y$ is a non-L-space

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For $Y$ is a closed, irreducible 3-manifold:

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Ozsváth-Szabó

Questions: How do we

(1) identify non-L-spaces?
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A non-trivial knot $K \subset S^3$ is an **L-Space knot** if $K$ admits a non-trivial surgery to an L-space.
Producing Non-L-Spaces via Dehn Surgery

Definition

A non-trivial knot $K \subset S^3$ is an **L-Space knot** if $K$ admits a non-trivial surgery to an L-space.

Examples: Torus knots
Suppose $K \subset S^3$. 
Suppose $K \subset S^3$. Then either:

1. $K$ isn’t an L-space knot:

2. $K$ is an L-space knot:

Theorem: (Kronheimer-Mrowka-Ozsváth-Szabó; J+S Rasmussen):

For $r \in \mathbb{Q}$, $S^3_r(K) = \begin{cases} 
\text{non-L-space} & r < 2g(K) - 1 \\
\text{L-space} & r \geq 2g(K) - 1 
\end{cases}$

If $r < 2g(K) - 1$, then $S^3_r(K)$ is always a non-L-space. Therefore, the LSC predicts it admits a taut foliation.
Suppose $K \subset S^3$. Then either:

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1. \textbf{$K$ isn’t an L-space knot:}
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Producing Non-L-Spaces via Dehn Surgery
Suppose $K \subset S^3$. Then either:

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Producing Non-L-Spaces via Dehn Surgery

Suppose $K \subset S^3$. Then either:

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   If $r < 2g(K) - 1$, then $S^3_r(K)$ is **always** a non-L-space.

   Therefore, the LSC predicts a taut foliation.
Theorem (K.)

Suppose $K \subset S^3$ is the closure of a positive 3-braid. Then for all rational $r < 2g(K) - 1$, $S^3_r(K)$ admits a taut foliation.
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Example: The $P(-2, 3, 7)$ (Fintushel-Stern) Pretzel Knot
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Suppose $K \subset S^3$ is the closure of a positive 3-braid. Then for all rational $r < 2g(K) - 1$, $S_r^3(K)$ admits a taut foliation.

Remark: Applying (Lidman-Moore '16), we get the first example of $Y$ is a non-L-space $\iff Y$ admits a taut foliation for every non-L-space obtained by Dehn surgery along an infinite family of hyperbolic L-space knots.
What’s Known, Revisited

For $Y$ is a closed, irreducible 3-manifold:

$Y$ admits a taut foliation $\Rightarrow$ $Y$ is a non-L-space

Ozsváth-Szabó

Questions: How do we
(1) identify non-L-spaces?
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A **branched surface** is a “co-oriented 2-complex”, locally modelled by: 

![Diagram of branched surfaces](image-url)
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Branched surfaces can encode data about

1. surfaces in 3-manifolds (Oertel; Floyd-Oertel)
2. laminations in 3-manifolds (Li)
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Proof Sketch:
(1) Identify a fiber surface for $K$ in $X_K = S^3 - \nu(K) = F \times I / \varphi$
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Branched Surfaces and Taut Foliations

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(3) Use $B$ to build a taut foliation in $X_K$
Branched Surfaces and Taut Foliations

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3. Use $B$ to build a taut foliation in $X_K$

4. Understand how $\mathcal{F}$ meets $\partial X_K$
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4. Understand how $\mathcal{F}$ meets $\partial X_K$
5. Dehn fill to produce a taut foliation in $S^3_r(K)$
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Branched surfaces give a recipe for building taut foliations.