SUPPLEMENTARY CALCULATIONS FOR A $\widetilde{GL}(4)$ EXPONENTIAL SUM

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1. Introduction

The goal of this document is the analysis of the exponential sum $S_{\ell,m}$ defined in (1) below, which arises in the computation of a maximal parabolic Whittaker coefficient on a $GL(4)$ metaplectic Eisenstein series. It is intended as a supplement to the paper [1] where some of these calculations, most notably the proof of Theorem 2.1 below, were omitted; in order to improve the readability of this document many statements have been transferred directly from [1]. For complete details about notation used and for the context in which this exponential sum arises, the reader should consult [1].

Let $n$ be a positive integer and $F$ a number field containing the $2n$-th roots of unity. Let $S$ be a finite set of places large enough so that $\mathfrak{o}_S$, the ring of $S$-integers, is a principal ideal domain. Let $\psi$ be a non-degenerate character of $F_S = \prod_{v \in S} F_v$ with conductor $\mathfrak{o}_S$. Throughout, we will fix a prime $p$ in $\mathfrak{o}_S$ and let $q$ denote the cardinality of $\mathfrak{o}_S/p\mathfrak{o}_S$.

Given a triple of $\mathfrak{o}_S$ integers $m = (m_1, m_2, m_3)$, we wish to analyze the following exponential sum for choices of integers $\ell = (\ell_1, \ell_2, \ell_3, \ell_4)$:

$$S_{\ell,m} := q^{2\ell_4} \sum \left( \frac{c_1}{p^{\ell_1}} \right) \left( \frac{c_2}{p^{\ell_2}} \right) \left( \frac{c_3}{p^{\ell_3}} \right) \left( \frac{c_4}{p^{\ell_4}} \right) \psi \left( -p^{m_1} \left( \frac{b_2c_1p^{\ell_1}}{p^{\ell_1+\ell_3}} + \frac{b_4c_3}{p^{\ell_3}} \right) + p^{m_2} \frac{c_4}{p^{\ell_4}} + p^{m_3} \left( \frac{c_1b_2p^{\ell_4}}{p^{\ell_1+\ell_2}} + \frac{c_3b_4}{p^{\ell_2}} \right) \right).$$

Here we sum over $c_i$ modulo $p^{\ell_i}$ for $i = 2, 3, 4$ and $c_1$ modulo $p^{\ell_1+\ell_2+\ell_3}$, with $c_i$ prime to $p$ if $\ell_i > 0$ and no such condition if $\ell_i = 0$; $b_i$ satisfies the congruence $b_i c_i \equiv 1 \pmod{p^{\ell_i}}$ for $i = 2, 3, 4$. The symbol $\left( \frac{\cdot}{q} \right)$ appearing in the sum is the $n$-th power residue symbol as in [1].

Because this sum arises in the Whittaker coefficient of an Eisenstein series, we have additional divisibility conditions:

$$\ell_1 + \ell_3 \leq m_1 + \ell_2 + \ell_4$$

(2)

$$\ell_1 + \ell_2 \leq m_3 + \ell_3 + \ell_4.$$  

(3)

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The results of the next section will show that, in fact, the following inequalities may be used to describe the vanishing of the exponential sum $S_{\ell,m}$:

\begin{align*}
\ell_4 &\leq m_2 + 1 \\
\ell_1 + \ell_2 &\leq m_3 + 1 + \min(\ell_1, \ell_4) \\
\ell_1 + \ell_3 &\leq m_1 + 1 + \min(\ell_1, \ell_4).
\end{align*}

2. Theorems on the Support of $\tilde{GL}_4$ Exponential Sums

The goal of the next two subsections is to prove the following theorem, which demonstrates that the exponential sum $S_{\ell,m}$ in our recurring $\tilde{GL}_4$ example is supported on the union of the $i_p$-Lusztig data for the elements of the finite crystal $B_{m+p}^\vee$ and an infinite collection of integer lattice points lying on a particular 2-dimensional hyperplane.

**Theorem 2.1.** The sum $S_{\ell,m}$ defined in (1) is zero unless either

\begin{enumerate}
\item the highest weight inequalities (4), (5), (6) hold, or
\item $\ell_1 = \ell_4 \leq m_2 + 1$ and $\ell_2 - m_3 = \ell_3 - m_1 > 1$.
\end{enumerate}

In the course of proving this theorem, we also provide an explicit evaluation of $S_{\ell,m}$ in case (b). In Section ??, we use this explicit formula to explain why the contribution from case (b) does not contribute to the Whittaker function. The evaluation of the sum in case (a) is postponed until Section ??.

2.1. Proof of Theorem 2.1: The Case of Positive Lusztig Data. In this section, we prove Theorem 2.1 in the case that all $\ell_i > 0$. This condition guarantees that each of the $c_i$ and $b_i$ are relatively prime to $p$. The exponential character is the product of three terms:

\[ \psi \left( -p^{m_1} \left( \frac{b_2 c_1 p^{\ell_4}}{p^{\ell_1 + \ell_3}} + \frac{b_1 c_3}{p^{\ell_3}} \right) \right) \cdot \psi \left( p^{m_2} \frac{c_4}{p^{\ell_4}} \right) \cdot \psi \left( p^{m_3} \left( \frac{c_1 b_3 p^{\ell_4}}{p^{\ell_1 + \ell_2}} + \frac{c_2 b_4}{p^{\ell_2}} \right) \right). \]

Consider the following change in the $c_2$ variable:

\[ c_2 \mapsto c_2 (1 + a p^{\max(\ell_2 - m_3, 1)}), \]

for any $a \mod p$. 


The third factor doesn’t change, and the power residue symbol is also unchanged. Hence analyzing the first term in the exponential, we obtain the following inequality as a necessary condition for the sum to be non-zero:

\[ \ell_1 + \ell_3 \leq m_1 + \ell_4 + \max(\ell_2 - m_3, 1). \] (8)

To analyze the inequality (8) further, we separate the following two cases for the value of \( \max(\ell_2 - m_3, 1) \).

**CASE A1:** \( \ell_2 \leq m_3 + 1 \). **CASE A2:** \( \ell_2 > m_3 + 1 \).

In Case A1, (8) gives \( \ell_1 + \ell_3 \leq m_1 + \ell_4 + 1 \). Note that in Case A1, (5) is satisfied if \( \min(\ell_1, \ell_4) = \ell_1 \), and (6) is satisfied if \( \min(\ell_1, \ell_4) = \ell_4 \).

In Case A2, let us make the change of variables

\[ c_2 \mapsto c_2(1 + ap^{\ell_2 - m_3 - 1}), \quad \text{for any } a \mod p. \]

Then \( b_2 \) gets replaced by \( b_2(1 - ap^{\ell_2 - m_3 - 1} + O(p^{2(\ell_2 - m_3 - 1)})) \). Then the dependence on \( a \) is given by

\[ \psi \left( \frac{ab_4c_2}{p} + \frac{ac_1b_2p^{m_1 + \ell_4 + \ell_2 - m_3 - 1}}{p^{\ell_1 + \ell_3}} \right) \]

times a term involving a larger power of \( p \) that will not change the subsequent analysis. As we sum over \( a \), such an exponential gives 0 unless both displayed terms have the same power of \( p \) in the denominator. This implies that the support of the sum in Case A2 satisfies \( \ell_1 + \ell_3 = m_1 + \ell_4 + \ell_2 - m_3 \).

The variables \( c_2 \) and \( c_3 \) have symmetric roles in the exponential sum. Thus we may perform analogous substitutions for \( c_3 \) and obtain the pair of conditions and their consequences when they are in the support of \( S_{\ell, m} \):

**CASE B1:** \( \ell_3 \leq m_1 + 1 \) \quad \implies \quad \ell_1 + \ell_2 \leq m_3 + \ell_4 + 1 \)

**CASE B2:** \( \ell_3 > m_1 + 1 \) \quad \implies \quad \ell_1 + \ell_2 = m_3 + \ell_4 + \ell_3 - m_1 \).

Symmetrically to the above, in Case B1, (6) is satisfied if \( \min(\ell_1, \ell_4) = \ell_1 \), and (5) is satisfied if \( \min(\ell_1, \ell_4) = \ell_4 \).

We now analyze various combinations of Cases A and B.

**Lemma 2.2.** Suppose that \( S_{\ell, m} \neq 0 \). If the inequality A1 holds, then B1 must hold. Similarly, if A2 holds, then B2 must hold.

**Proof.** The two cases are symmetric; we do the second. Suppose that A2 and B1 hold and \( S_{\ell, m} \neq 0 \). Then we have the conditions

\[ \ell_2 > m_3 + 1 \quad \text{and} \quad \ell_1 + \ell_3 = m_1 + \ell_4 + \ell_2 - m_3 \]

\[ \ell_3 \leq m_1 + 1 \quad \text{and} \quad \ell_1 + \ell_2 \leq m_3 + \ell_4 + 1. \]

The conditions \( \ell_2 > m_3 + 1 \) and \( \ell_1 + \ell_2 \leq m_3 + \ell_4 + 1 \) imply that \( \ell_1 < \ell_4 \). But \( \ell_3 \leq m_1 + 1 \), so this implies that \( \ell_1 + \ell_3 < \ell_4 + m_1 + 1 \). Combining this with \( \ell_1 + \ell_3 = m_1 + \ell_4 + \ell_2 - m_3 \) gives \( \ell_2 \leq m_3 \), a contradiction. \( \square \)

**Lemma 2.3.** If the A1 and B1 inequalities hold and the sum \( S_{\ell, m} \neq 0 \), then the highest weight inequalities (4), (5), (6) hold.
Proof. The dependence of the character $\psi$ on $b_4, c_4$ is given by
\[ \psi\left(p^{m_2-\ell_4}c_4 + b_4\left(-c_3p^{m_1-\ell_3} + c_2p^{m_3-\ell_2}\right)\right). \]

The hypotheses guarantee that $\text{ord}_p(-c_3p^{m_1-\ell_3} + c_2p^{m_3-\ell_2}) \geq -1$. But the sum
\[ \sum_{c_4 \mod p^{\ell_4}, (c_4,p)=1} \left(\frac{c_4}{p^{\ell_4}}\right) \psi\left(\frac{c_4}{p^{\ell_4}} + \frac{b_4}{p^{1-s}}\right) \]
is zero for any $r, s \geq 0$, as the variable change $c_4 \mapsto c_4(1 + ap^{1+r})$ readily shows. Hence we must have $\ell_4 \leq m_2 + 1$ in order for the sum over $c_4$ to be non-vanishing. Under the A1, B1 inequalities, (5) and (6) also hold, since no matter which of $\ell_1, \ell_4$ is their minimum, one of these inequalities follows from A1 and the other from B1. \hfill \square

Lemma 2.4. If the A2 and B2 inequalities hold, then the exponential sum $S_{\ell,m}$ vanishes unless
\[ \ell_1 = \ell_4, \quad m_1 + \ell_2 = m_3 + \ell_3. \] (9)

In that case, using the notation in (7),
\[ S_{\ell,m} = q^{\ell_2 + \ell_3 + 2\ell_4 - k - \ell_1} g_0(m_2, \ell_4) h(2\ell_1 + \ell_2 + \ell_3) \] (10)
with $k = \ell_2 - m_3 = \ell_3 - m_1$ (and $g_0(m_2, 0) = 1$ by definition).

Proof. We consider the possible support of $S_{\ell,m}$ in this case. Since $\ell_1 + \ell_3 = m_1 + \ell_4 + \ell_2 - m_3$ and $\ell_1 + \ell_2 = m_3 + \ell_4 + \ell_3 - m_1$, the equalities (9) follow at once. Then the dependence of the summand on $c_1$ is given by
\[ \left(\frac{c_1}{p^{\ell_1}}\right) \psi\left(-p^{m_1}\left(\frac{b_2c_1p^{\ell_4}}{p^{\ell_1+\ell_4}}\right) + p^{m_3}\left(\frac{c_1b_3p^{\ell_4}}{p^{\ell_1+\ell_4}}\right)\right) = \left(\frac{c_1}{p^{\ell_1}}\right) \psi(p^{-k}(b_2 - b_3)c_1), \]
with $k > 1$ as defined in the statement of the lemma. Thus the sum over $c_1$ will be 0 unless $b_2 \equiv b_3 \mod p^{k-1}$. Note that
\[ \sum_{c_1} \left(\frac{c_1}{p^{\ell_1}}\right) \psi(p^{-k}(b_2 - b_3)c_1) \]
is equal to
\[ q^{\ell_2 + \ell_3} \left(\frac{u^{-1}}{p^{\ell_1}}\right) g(\ell_1) \]
if $b_2 - b_3 = up^{k-1}$ with $(u,p) = 1$, and instead gives $q^{\ell_2 + \ell_3} h(\ell_1)$ if $p^k | b_2 - b_3$. Thus the total remaining sum is:
\[ q^{\ell_2 + \ell_3 + 2\ell_4} g(\ell_1) \sum_{b_2-b_3=up^{k-1}} \left(\frac{u^{-1}}{p^{\ell_1}}\right) \left(\frac{c_2}{p^{\ell_2}}\right) \left(\frac{c_3}{p^{\ell_3}}\right) \left(\frac{c_4}{p^{\ell_4}}\right) \psi\left(\frac{b_4(c_2 - c_3)}{p^{k+1}} + \frac{c_4p^{m_2}}{p^{\ell_4}}\right) \]
\[ + q^{\ell_2 + \ell_4 + 2\ell_4} h(\ell_1) \sum_{c_2,c_3,c_4} \left(\frac{c_2}{p^{\ell_2}}\right) \left(\frac{c_3}{p^{\ell_3}}\right) \left(\frac{c_4}{p^{\ell_4}}\right) \psi\left(\frac{c_4p^{m_2}}{p^{\ell_4}}\right). \]

We note that since $k > 1$, in both sums $b_2 \equiv b_3 \mod p$ and so $(c_3/p^{\ell_3}) = (c_2/p^{\ell_3})$. Thus the second term gives
\[ q^{\ell_2 + \ell_4 + 2\ell_4} h(\ell_1) g(m_2, \ell_4) h(\ell_2 + \ell_3). \] (11)
For the first term, note that if \( b_2 - b_3 = up^{k-1} \) then \( c_2 - c_3 = b_2^{-1} - b_3^{-1} = -c_2c_3up^{k-1} \) modulo \( p^k \). Regard the sum as over \( c_4, c_2 \) and \( u \), and make a change of variables \( u \mapsto ub_2^3c_4 \). Then the first term gives a contribution of

\[
q^{\ell_2+\ell_3+2\ell_4}g(\ell_1) \sum_{u \mod p^{\ell_1+\ell_3+1}} \left( \frac{u - c_2c_3^{-1}}{p^{\ell_1}} \right) \left( \frac{c_2}{p^{\ell_2+\ell_3}} \right) \left( \frac{c_4}{p^{\ell_4}} \right) \psi \left( \frac{-u + c_4p^{m_2}}{p^{\ell_4}} \right). \tag{12}
\]

The three sums now separate and they evaluate as follows. The \( c_2 \) sum gives \( q^{-2\ell_1-\ell_3}h(2\ell_1 + \ell_2 + \ell_3) \). The sum over \( u \) gives \( q^{2-\ell_4-\ell_1-\ell_3} \) times the conjugate of \( g(\ell_1) \). Since \( \ell_1 = \ell_4 \), the multiplicative characters in \( c_4 \) cancel, and then the \( c_4 \) sum gives \( g_0(m_2, \ell_4) \). Thus this term contributes

\[
q^{\ell_2+\ell_3+2\ell_4-\ell_1-\ell_3}h(2\ell_1 + \ell_2 + \ell_3)g_0(m_2, \ell_4)
\]

if \( n \) does not divide \( \ell_1 \) and

\[
q^{\ell_2+\ell_3+2\ell_4-\ell_1-\ell_3}h(2\ell_1 + \ell_2 + \ell_3)g_0(m_2, \ell_4)
\]

if \( n \mid \ell_1 \). Adding (11) to the expressions above gives the result. (Note that when \( n \mid \ell_1 = \ell_4 \), one has \( g(m_2, \ell_4) = g_0(m_2, \ell_4) \).)

This concludes the proof of Theorem 2.1 in the case when all the Lusztig data \( \ell_i \) are positive.

2.2. Proof of Theorem 2.1: Remaining Cases. We turn to a consideration of Theorem 2.1 when at least one of the \( \ell_i = 0 \).

Suppose \( \ell_2 = 0 \). Then we may take \( b_2 = c_2 = 0 \) and remove \( (c_2/p^{\ell_2}) \). The sum becomes

\[
q^{2\ell_4} \sum \left( \frac{c_1}{p^{\ell_1}} \right) \left( \frac{c_2}{p^{\ell_2}} \right) \left( \frac{c_4}{p^{\ell_4}} \right) \psi \left( -p^{m_1} \frac{b_4c_3}{p^{\ell_3}} + p^{m_2} \frac{c_4}{p^{\ell_4}} + p^{m_3} \frac{c_1b_4p^{\ell_4}}{p^{\ell_1}} \right).
\]

If in addition \( \ell_3 = 0 \), we may take \( b_3 = c_3 = 0 \) and remove \( (c_3/p^{\ell_3}) \). The divisibility conditions (2), (3) become \( \ell_1 \leq \min(m_1, m_3) + \ell_4 \). If \( \ell_4 = 0 \) we are done. If \( \ell_4 \neq 0 \), then we obtain a sum over \( c_4 \) which vanishes unless \( \ell_4 \leq m_2 + 1 \). So we obtain all highest weight inequalities for \( \ell_2, \ell_3 = 0 \).

If \( \ell_2 = 0 \) but \( \ell_3 \neq 0 \), then \( (b_3, p) = 1 \). The sum over \( c_1 \) vanishes unless \( \ell_1 \leq m_3 + \ell_4 + 1 \). If \( \ell_4 = 0 \) then we also have the divisibility condition \( \ell_1 + \ell_3 \leq m_1 \). These give the highest weight inequalities.

If \( \ell_2 = 0, \ell_3, \ell_4 \neq 0 \), we make the changes \( c_1 \mapsto c_1c_3 \) and then \( c_3 \mapsto c_3c_4 \). Then the sum factors. The \( c_1 \) sum gives \( q^{\ell_1}g(m_3 + \ell_4, \ell_1) \), the \( c_3 \) sum gives \( q^{-\ell_1}g(\ell_1 + m_1, \ell_1 + \ell_3) \), and the \( c_4 \) sum gives \( q^{-\ell_1-\ell_3}g(\ell_1 + \ell_3 + m_2, \ell_1 + \ell_3 + \ell_4) \). These sums vanish unless \( \ell_3 \leq m_1 + 1 \) and \( \ell_4 \leq m_2 + 1 \). Once again, the desired inequalities follow after incorporating the divisibility conditions. This completes all cases with \( \ell_2 = 0 \). The cases with \( \ell_3 = 0 \) are symmetric.

Suppose now \( \ell_4 = 0 \) but \( \ell_2, \ell_3 \neq 0 \). Since \( \ell_4 = 0 \), we may take \( b_4 = c_4 = 0 \) and the sum \( S_{\ell,m} \) simplifies to:

\[
q^{2\ell_4} \sum \left( \frac{c_1}{p^{\ell_1}} \right) \left( \frac{c_2}{p^{\ell_2}} \right) \left( \frac{c_3}{p^{\ell_3}} \right) \psi \left( -p^{m_1} \frac{b_2c_1}{p^{\ell_1+\ell_3}} + p^{m_3} \frac{c_1b_3}{p^{\ell_1+\ell_3}} \right).
\]
If \( c_1 \not\equiv 0 \pmod{p} \), then we may sum over \( b_2, b_3 \) which are relatively prime to \( p \). This sum is zero unless the following inequalities hold:

\[
\ell_1 + \ell_3 < m_1 + 1, \quad \ell_1 + \ell_2 \leq m_3 + 1.
\]  

These imply the desired highest weight inequalities. This condition on \( c_1 \) is guaranteed whenever \( \ell_1 > 0 \). Thus we have established Theorem 2.1 in this case.

It remains to analyze the case when \( \ell_1, \ell_4 = 0 \), but \( \ell_2, \ell_3 \neq 0 \). As we have shown above, the summands with \( (c_1, p) = 1 \) do not contribute to the value of the sum except when the highest weight inequalities are satisfied. However, this is not sufficient because in the Iwasawa decomposition that gives rise to the sum, summands with \( (c_1, p) = p \) do occur when \( \ell_1 = 0 \). Accordingly, we now examine the sum over such \( c_1 \) more closely. (Note that if \( \ell_1 = 0 \) then the \( n \)-th power residue symbol in \( c_1 \) is identically 1.)

Let \( \text{ord}_p(c_1) = j \) with \( j \geq 1 \) and write \( c_1 = p^jc_1' \). Substituting into the sum, the contribution becomes

\[
\sum \left( \frac{c_2}{p^{\ell_2}} \right) \left( \frac{c_3}{p^{\ell_3}} \right) \psi \left( -p^{m_1+j} \left( \frac{b_2c_1'}{p^{\ell_2}} \right) + p^{m_3+j} \left( \frac{c_1'b_3}{p^{\ell_3}} \right) \right),
\]

where the sum is over \( c_1 \) mod \( p^{\ell_i} \), \( (c_1, p) = 1 \), \( i = 2, 3 \), and over \( c_1' \) modulo \( p^{\ell_2+\ell_3-j} \) with \( (c_1', p) = 1 \). We change \( c_2 \mapsto c_2c_1' \) and \( c_3 \mapsto c_3c_1' \); this removes \( c_1' \) from the argument of \( \psi \) and introduces the power residue symbol \( (c_1'/p^{\ell_2+\ell_3}) \). The sum over \( c_2 \), resp. \( c_3 \), contributes

\[
\begin{cases}
    h(\ell_2) & \text{if } m_1 + j \geq \ell_3, \\
    \bar{g}(\ell_2) & \text{if } m_1 + j = \ell_3 - 1, \\
    0 & \text{otherwise},
\end{cases}
\]

\[
\begin{cases}
    h(\ell_3) & \text{if } m_3 + j \geq \ell_2, \\
    \bar{g}(\ell_3) & \text{if } m_3 + j = \ell_2 - 1, \\
    0 & \text{otherwise}.
\end{cases}
\]

Here \( \bar{g} \) is the Gauss sum made from the conjugate \( n \)-th power residue symbol. Note that if \( n \) divides both \( \ell_2 \) and \( \ell_3 \), these values are:

\[
\begin{cases}
    \phi(p^{\ell_2}) & \text{if } m_1 + j \geq \ell_3, \\
    -q^{\ell_2-1} & \text{if } m_1 + j = \ell_3 - 1, \\
    0 & \text{otherwise},
\end{cases}
\]

\[
\begin{cases}
    \phi(p^{\ell_3}) & \text{if } m_3 + j \geq \ell_2, \\
    -q^{\ell_3-1} & \text{if } m_3 + j = \ell_2 - 1, \\
    0 & \text{otherwise}.
\end{cases}
\]

Suppose that \( \ell_3 - m_1 < \ell_2 - m_3 \). The highest weight inequalities are satisfied unless \( \ell_2 - m_3 > 1 \). In that case, the sum over \( j, 1 \leq j \leq \ell_2 + \ell_3 \), gives

\[
h(\ell_2) \left( \bar{g}(\ell_3) h(\ell_2 + \ell_3) q^{-\ell_2 + m_1 + 1} + h(\ell_3) \left( 1 + \sum_{j=\ell_2-m_3}^{\ell_2+\ell_3-1} h(\ell_2 + \ell_3) q^{-j} \right) \right). \tag{14}
\]

Because of the presence of the functions \( h \), the expression (14) is zero unless \( n \) divides both \( \ell_2 \) and \( \ell_3 \). In that case, (14) is given by

\[
\phi(p^{\ell_2}) \left( (-q^{\ell_3-1}) \phi(p^{\ell_3+m_1+1}) + \phi(p^{\ell_3}) \sum_{j=\ell_2-m_3}^{\ell_2+\ell_3-1} \phi(p^{\ell_2+\ell_3-j}) \right) = 0.
\]

Similarly the sum is zero if \( \ell_3 - m_1 > \ell_2 - m_3 \).

This proves the description of the support of \( S_{\ell,m} \) given in Theorem 2.1. In the one case where the highest weight inequalities are not satisfied, that is, in the case \( \ell_3 - m_1 = \ell_2 - m_3 > 1 \).
1, we obtain the contribution
\[ \bar{g}(\ell_2) \bar{g}(\ell_3) h(\ell_2 + \ell_3) q^{-\ell_2 + m_3 + 1} + h(\ell_2) h(\ell_3) \left\{ 1 + \sum_{j=\ell_2 - m_3}^{\ell_2 + \ell_3 - 1} h(\ell_2 + \ell_3) q^{-j} \right\}. \]

This is zero unless:
(1) \( n \) does not divide \( \ell_2 \) and \( \ell_3 \) but \( n \) divides \( \ell_2 + \ell_3 \); or
(2) \( n \) divides \( \ell_2 \) and \( \ell_3 \).

In case (1), the sums \( \bar{g}(\ell_2) \) and \( \bar{g}(\ell_3) \) are (up to a power of \( q \)) nontrivial conjugate Gauss sums; thus in this case the contribution is \( q^{m_3 + \ell_3} \phi(p^{\ell_2 + \ell_3}) \).

In case (2), the contribution is

\[ q^{\ell_2 + \ell_3 - 2} \phi(p^{\ell_2 + m_3 + 1}) + \phi(p^{\ell_2}) \phi(p^{\ell_3}) \sum_{j=\ell_2 - m_3}^{\ell_2 + \ell_3} \phi(p^{\ell_2 + \ell_3 - j}). \]

Since the sum telescopes, this simplifies to

\[ q^{\ell_2 + \ell_3 - 2} \phi(p^{\ell_2 + m_3 + 1}) + q^{\ell_3 + m_3} \phi(p^{\ell_2}) \phi(p^{\ell_3}) = \phi(p^{\ell_2 + \ell_3 + m_3 + \ell_3}). \]  \hspace{1cm} (15)

Note that this is equal to the contribution in case (1). Since \( \ell_1 = \ell_4 = 0 \), this expression agrees with (10). This completes the proof of Theorem 2.1.

Combining the result of (15) together with Lemma 2.4, we also obtain:

**Proposition 2.5.** In case (b) of Theorem 2.1, with notation as in (7),
\[ S_{\ell,m} = q^{\ell_2 + \ell_3 + 2\ell_4 - k - \ell_1} g_0(m_2, \ell_4) h(2\ell_1 + \ell_2 + \ell_3) \] \hspace{1cm} (16)

with \( k = \ell_2 - m_3 = \ell_3 - m_1 \) (and \( g_0(m_2, 0) = 1 \) by definition).

**References**


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