IMAGES OF THE DISK COMPLEX

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Abstract. Let M be an orientable irreducible 3–manifold with boundary and let S be a subsurface of \( \partial M \). Suppose every curve in \( \partial S \) is disk-busting, i.e. every curve in \( \partial S \) intersects every compressing disk in M. We show that either (1) M is an I–bundle of which S is a horizontal boundary component or (2) under a natural projection, the image of the disk complex of M has diameter at most 12 in the curve complex of S.

1. Introduction

Let \( F \) be a closed and orientable surface of genus at least 2. The curve complex of \( F \), first defined by Harvey [1], is the complex whose vertices are the isotopy classes of essential simple closed curves in \( F \), and \( k + 1 \) vertices determine a \( k \)-simplex if they are represented by pairwise disjoint curves. We denote the curve complex of \( F \) by \( \mathcal{C}(F) \). For any two vertices in \( \mathcal{C}(F) \), one can define the distance \( d(x, y) \) to be the minimal number of 1–simplices in a simplicial path jointing \( x \) to \( y \) over all such possible paths.

The definition of the curve complex can be extended to torus and surfaces with boundary. In all cases, the vertices of the curve complex of \( F \) are the isotopy classes of essential non-peripheral simple closed curves in \( F \). The definition of \( \mathcal{C}(F) \) for such surfaces is the same as above except for a few sporadic cases. If \( F \) is an annulus or a pair of pants, as every non-trivial simple closed curve in \( F \) is peripheral, \( \mathcal{C}(F) \) is empty. If \( F \) is a torus or a once-punctured torus or a sphere with 4 punctures, then edges are placed between vertices corresponding to curves of smallest possible intersection number (the resulting \( \mathcal{C}(F) \) is the Farey graph). The structure of the curve complex has been extensively studied by H. Masur and Y. Minsky. For example, they proved that \( \mathcal{C}(F) \) is \( \delta \)-hyperbolic [6].

Let \( S \) be a compact orientable surface with boundary and suppose \( S \) is not a disk or an annulus. The arc-and-curve complex \( \mathcal{AC}(S) \) is defined as follows: Each vertex of \( \mathcal{AC}(S) \) is the isotopy class of either an essential properly embedded arc or an essential non-peripheral simple closed curve in \( S \), and a set of vertices forms a simplex of \( \mathcal{AC}(S) \) if these vertices are represented by pairwise disjoint arcs or curves in \( S \). Similar to the curve complex, the arc-and-curve complex is also very useful in the study of mapping class groups. In this paper, we are mainly interested in the vertices of these complexes and the simplicial distance between the vertices as defined above.

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Suppose $F$ is a closed surface and $S$ is a compact essential non-annular subsurface of $F$, in particular, $\partial S$ consists of essential curves in $F$. Then, for any subcomplex $\mathcal{D}$ of $\mathcal{C}(F)$, there is a natural projection $\pi_A: \mathcal{D} \to \mathcal{AC}(S)$ defined as follows: For every vertex $[\gamma]$ in $\mathcal{D}$, we take a curve $\gamma$ in the isotopy class so that $|\gamma \cap S|$ is minimal. If $\gamma \cap S = \emptyset$, we set $\pi_A([\gamma]) = \emptyset$. If $\gamma \cap S \neq \emptyset$, then $\pi_A([\gamma])$ is the isotopy class of a component of $\gamma \cap S$.

Note that $\gamma \cap S$ may have more than one component, but we can choose an arbitrary component for $\pi_A([\gamma])$. Furthermore, there is also a natural projection $\pi_0$ from the vertices of $\mathcal{AC}(S)$ to $\mathcal{C}(S)$ which is the identity map for closed curves and for each essential arc $\alpha$, $\pi_0([\alpha])$ is the isotopy class of a non-peripheral boundary component of $N(\alpha \cup \partial S)$, where $N(\alpha \cup \partial S)$ is a small neighborhood of $\alpha \cup \partial S$ in $S$. We define the projection $\pi_C: \mathcal{D} \to \mathcal{C}(S)$ to be $\pi_C = \pi_0 \circ \pi_A$. Such projections were used by H. Masur and Y. Minsky in their study of the structure of the curve complex.

Let $M$ be a compact orientable and irreducible 3–manifold and $F$ a component of $\partial M$. We suppose $F$ is the only compressible boundary component of $M$. Let $D$ be the subcomplex of $\mathcal{C}(F)$ with each vertex of $D$ corresponding to a curve that bounds a compressing disk in $M$. We call $D$ the disk complex for $F$. If $M$ is a handlebody, then the disk complex is particularly interesting, for example see [2]. H. Masur and Y. Minsky proved that the disk complex of a handlebody is quasi-convex in the curve complex of $\partial M$ [8].

A simple closed curve $\Gamma$ in $F$ is said to be disk-busting if every compressing disk in $M$ intersects $\Gamma$. In terms of the curve complex, a disk-busting curve is a curve representing a vertex of distance at least 2 from the disk complex. For example, if $N = H_1 \cup_S H_2$ is a strongly irreducible Heegaard splitting, then the boundary of every compressing disk in $H_2$ is a disk-busting curve in the Heegaard surface for $H_1$.

Let $S$ be an essential compact non-annular subsurface of $F$. Then, as above, we can define projections $\pi_A$ and $\pi_C$ from the disk complex $D$ to $\mathcal{AC}(S)$ and $\mathcal{C}(S)$ respectively.

For any $I$–bundle $J$ over a compact surface $P$ with $\partial P \neq \emptyset$, one can divide $\partial J$ into two parts: (1) the vertical boundary $\partial_v J$ which is the $I$–bundle restricted to $\partial P$, and (2) the horizontal boundary $\partial_h J = \partial J - \partial_v J$ which is the portion of $\partial J$ transverse to the $I$–fibers. Note that if $J$ is orientable, $\partial_v J$ is the collection of annuli $\partial P \times I$.

Clearly if $S$ is a component of the horizontal boundary $\partial_h J$ of an $I$–bundle $J$, then the images $\pi_A(D)$ and $\pi_C(D)$ of the disk complex (for $\partial J$) can be unbounded in $\mathcal{AC}(S)$ and $\mathcal{C}(S)$ respectively. We show in this paper that in other cases, the images of the disk complex are bounded by a specific number.

**Theorem 1.** Let $M$ be a compact orientable and irreducible 3–manifold and $F$ a component of $\partial M$. Suppose $\partial M - F$ is incompressible in $M$. Let $\mathcal{D}$ be
the disk complex for $F$. Let $S$ be a compact connected subsurface of $F$ and suppose every component of $\partial S$ is disk-busting. Then either

1. $M$ is an $I$–bundle over a compact surface, $S$ is a component of the horizontal boundary of this $I$–bundle, and the vertical boundary of this $I$–bundle is a single annulus, or

2. the image $\pi_A(D)$ of the disk complex lies in a ball of radius 3 in $\mathcal{AC}(S)$, in particular, $\pi_A(D)$ has diameter at most 6 in $\mathcal{AC}(S)$. Moreover $\pi_C(D)$ has diameter at most 12 in $C(S)$.

Corollary 2. Let $M$ be a compact orientable and irreducible 3–manifold and $F$ a component of $\partial M$. Suppose $\partial M - F$ is incompressible in $M$. Let $S_1$ and $S_2$ be two homeomorphic essential subsurfaces of $F$. Suppose each component of $\partial S_i$ is disk-busting. Then there are vertices $v_1$ and $v_2$ in $\mathcal{AC}(S_1)$ and $\mathcal{AC}(S_2)$ respectively, such that, for any homeomorphism $\phi: S_1 \to S_2$ satisfying $d(\phi(v_1), v_2) > 12$ in $\mathcal{AC}(S_2)$, the manifold obtained by gluing $S_1$ to $S_2$ via $\phi$ has incompressible boundary.

The proof of this corollary is straightforward. If the resulting manifold is compressible, then after isotopy, a compressing disk of the resulting manifold comes from a gluing of compressing disks in $M$, which means that the images of the disk complex of $M$ in $\mathcal{AC}(S_1)$ and $\mathcal{AC}(S_2)$ must overlap under the gluing map $\phi$. Let $v_1$ and $v_2$ be vertices of $\mathcal{AC}(S_1)$ and $\mathcal{AC}(S_2)$ respectively lying in images of the disk complex of $M$. By Theorem 1 if $d(\phi(v_1), v_2) > 12$, then the two images of the disk complex of $M$ do not overlap under the gluing map $\phi$. So the resulting manifold has no compressing disk.

Theorem 3 follows immediately from the following theorem, where we weaken the condition on $\partial S$.

Theorem 3. Let $M$ be a compact orientable and irreducible 3–manifold and $F$ a component of $\partial M$. Suppose $\partial M - F$ is incompressible in $M$. Let $D$ be the disk complex for $F$. Let $S$ be a compact connected subsurface of $F$ and suppose $F - \partial S$ is incompressible in $M$. Then either

1. there is a (possibly empty) collection of properly embedded incompressible annuli $A$ in $M$ which decomposes $M$ into submanifolds $J$ and $M'$ ($M' = \emptyset$ if $A = \emptyset$), such that
   (a) $J$ is an $I$–bundle over a compact connected surface and $S$ is a component of the horizontal boundary of $J$
   (b) each component of $A$ is a component of the vertical boundary $\partial_v J$ of $J$, and $\partial_v J$ has at least one more component lying in $F$. or

2. the image $\pi_A(D)$ of the disk complex lies in a ball of radius 3 in $\mathcal{AC}(S)$, in particular, $\pi_A(D)$ has diameter at most 6 in $\mathcal{AC}(S)$. Moreover $\pi_C(D)$ has diameter at most 12 in $C(S)$.

Theorem 1 follows from Theorem 3 because if $M' \neq \emptyset$ in part (1) of Theorem 3 (or if $M' = \emptyset$ but $\partial_v J$ has more than one component), then $M$ has a compressing disk totally in $J$ and disjoint from $A$ (or a component of
\(\partial_c J\), which means that a curve in \(\partial S\) is not disk-busting though \(F - \partial S\) is incompressible. Hence this paper is devoted to proving Theorem 3.

A note on the main theorem was originally written in 2005. The author decided to publish this paper after finding interesting applications of the main theorem and after learning that this result is not well-known even among some experts. Indeed, the main theorem is a crucial ingredient in the author’s program of studying amalgamation of Heegaard splittings.

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The main theorem, without the specific bound, was also proved by H. Masur and S. Schleimer independently but earlier. H. Masur and S. Schleimer use this theorem to study the so-called holes in the disk complex of a handlebody. The author would like to thank S. Schleimer for several email communications which led the author to realize that the special case originally considered by the author can be trivially extended to the full version of the main theorem.

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2. A relation between \(\mathcal{AC}(S)\) and \(\mathcal{C}(S)\)

**Notation.** Throughout this paper, for any topological space \(X\), we use \(\text{int}(X), \overline{X}\) and \(|X|\) to denote the interior, the closure and the number of components of \(X\) respectively. Given any subset \(X\) of a topological space \(Y\), the frontier of \(X\) in \(Y\) is the set of points in the closure of both \(X\) and \(Y - X\). So if \(X\) is a subsurface of the surface \(S\), then the frontier of \(X\) in \(S\) is a collection of properly embedded arcs and curves. We use \(N(X)\) to denote (the closure of) a small regular neighborhood of \(X\) in \(Y\).

Throughout this paper, to simplify notation, we do not distinguish between a vertex in \(\mathcal{AC}(S)\) or \(\mathcal{C}(S)\) and a well-chosen arc or curve that represents this vertex.

In this section, we give a simple relation between \(\mathcal{AC}(S)\) and \(\mathcal{C}(S)\) via the projection \(\pi_0: \mathcal{AC}(S) \rightarrow \mathcal{C}(S)\).

**Lemma 2.1.** Let \(S\) be a compact orientable surface. Let \(u\) and \(v\) be any two vertices of distance \(k\) in \(\mathcal{AC}(S)\). Then \(d(\pi_0(u), \pi_0(v)) \leq 2k\) in \(\mathcal{C}(S)\).

**Proof.** We may assume that \(S\) is not an annulus or a pair of pants, since the curve complex is empty in these two cases.

Given two arcs or closed curves \(\alpha\) and \(\beta\) in \(S\) of distance one in \(\mathcal{AC}(S)\), i.e., \(\alpha \cap \beta = \emptyset\) and \(\alpha\) is not parallel to \(\beta\), the map \(\pi_0\) (see section 1 for the definition of \(\pi_0\)) sends them to a pair of closed curves in \(S\).

**Claim.** \(d(\pi_0(\alpha), \pi_0(\beta)) \leq 2\) in \(\mathcal{C}(S)\).
Proof of the Claim. First suppose $\alpha$ is a closed curve, then by the definition of $\pi_0$, $\pi_0(\alpha) = \alpha$. Since $\alpha \cap \beta = \emptyset$, the curve $\pi_0(\beta)$ is disjoint from $\alpha = \pi_0(\alpha)$ and hence $d(\pi_0(\alpha), \pi_0(\beta)) \leq 1$ in $C(S)$. Thus, to prove the claim, we may assume both $\alpha$ and $\beta$ are properly embedded essential arcs in $S$. Recall that $\pi_0(\alpha)$ and $\pi_0(\beta)$ are chosen to be non-peripheral boundary components of $N(\partial S \cup \alpha)$ and $N(\partial S \cup \beta)$ respectively. So both curves $\pi_0(\alpha)$ and $\pi_0(\beta)$ lie in $N(\partial S \cup \alpha \cup \beta)$.

Now we consider the subsurface $N(\partial S \cup \alpha \cup \beta)$. As $\alpha \cap \beta = \emptyset$, the Euler characteristic of $N(\partial S \cup \alpha \cup \beta)$ is $-2$. Thus if $\chi(S) < -2$, then there is a closed essential and non-peripheral curve $\gamma$ of $S$ lying in $S - N(\partial S \cup \alpha \cup \beta)$. Since both $\pi_0(\alpha)$ and $\pi_0(\beta)$ lie in $N(\partial S \cup \alpha \cup \beta)$, this implies that $d(\pi_0(\alpha), \pi_0(\beta)) \leq d(\pi_0(\alpha), \gamma) + d(\gamma, \pi_0(\beta)) \leq 2$ in $C(S)$. So to prove the claim, it remains to consider the case that $\chi(S) \geq -2$. Since we have assumed $S$ is not an annulus or a pair of pants, $\chi(S) \geq -2$ implies that $S$ is either a once-punctured torus, a twice-punctured torus, or a 4-hole sphere.

Case (a). $S$ is a once-punctured torus.

As $S$ is a once-punctured torus, $\alpha$ and $\beta$ are non-separating arcs in $S$. Since $\alpha \cap \beta = \emptyset$ and since $\alpha$ and $\beta$ are not parallel in $S$, the endpoints of $\alpha$ and $\beta$ alternate along $\partial S$, which implies that the intersection number of the closed curves $\pi_0(\alpha)$ and $\pi_0(\beta)$ in $S$ is one. The curve complex for $S$ in this case is the Farey graph and $d(\pi_0(\alpha), \pi_0(\beta)) = 1$.

Case (b). $S$ is a twice-punctured torus.

We have the following subcases to consider.

If an arc, say $\alpha$, is a separating arc in $S$, then a component of $S - N(\alpha)$ is a once-punctured torus. Hence $S - N(\alpha \cup \beta)$ contains a simple closed curve $\gamma$ which is non-separating in $S$ and disjoint from both $\alpha$ and $\beta$. This implies that $\gamma$ is disjoint from both $\pi_0(\alpha)$ and $\pi_0(\beta)$. Thus we have $d(\pi_0(\alpha), \pi_0(\beta)) \leq d(\pi_0(\alpha), \gamma) + d(\gamma, \pi_0(\beta)) \leq 2$ in $C(S)$.

Next we may assume both $\alpha$ and $\beta$ are non-separating arcs. If an arc, say $\alpha$, connects the two boundary circles of $S$, then $S - N(\alpha)$ is a once-punctured torus. Hence $S - N(\alpha \cup \beta)$ contains a simple closed curve $\gamma$ which is non-separating in $S$ and disjoint from both $\alpha$ and $\beta$. As in the subcase above, we have $d(\pi_0(\alpha), \pi_0(\beta)) \leq d(\pi_0(\alpha), \gamma) + d(\gamma, \pi_0(\beta)) \leq 2$. So we may suppose next that neither $\alpha$ nor $\beta$ connects the two circles in $\partial S$.

If $\partial \alpha$ lies in one boundary circle and $\partial \beta$ lies in the other boundary circle of $S$, then $S - N(\alpha \cup \beta)$ is a pair of annuli and the core of either annulus is a non-separating curve $\gamma$ in $S$ disjoint from $\alpha \cup \beta$. As above, in this subcase we also have $d(\pi_0(\alpha), \pi_0(\beta)) \leq d(\pi_0(\alpha), \gamma) + d(\gamma, \pi_0(\beta)) \leq 2$ in $C(S)$.

The remaining subcase is that all 4 endpoints of $\alpha$ and $\beta$ lie in the same boundary circle of $S$. This subcase is similar to the case that $S$ is a once-puncture torus. Since we have assumed both $\alpha$ and $\beta$ are non-separating arcs in $S$, in this subcase, $\pi_0(\alpha)$ and $\pi_0(\beta)$ are a pair of non-separating closed curves intersecting in a single point. This means that $N(\pi_0(\alpha) \cup \pi_0(\beta))$ is
a once-punctured torus and its complement in $S$ is a pair of pants. The boundary circle $\gamma$ of $N(\pi_0(\alpha) \cup \pi_0(\beta))$ is an essential and non-peripheral curve in $S$. So $d(\pi_0(\alpha), \pi_0(\beta)) \leq d(\pi_0(\alpha), \gamma) + d(\gamma, \pi_0(\beta)) \leq 2$ in $C(S)$.

Case (c). $S$ is a 4-hole sphere.

In the curve complex of a 4-hole sphere, two closed curves have distance one if and only if they intersect in two points and the curve complex is the same as the Farey graph.

Let $\Gamma_\alpha$ be the union of the boundary circles of $S$ containing the endpoints of $\alpha$. If $\beta$ has at least one endpoint in $\partial S - \Gamma_\alpha$, then it follows from the definition of $\pi_0$ that $\beta \cap \pi_0(\alpha)$ has at most one intersection point, which implies that $\pi_0(\beta) \cap \pi_0(\alpha)$ contains at most 2 intersection points. Thus $d(\pi_0(\alpha), \pi_0(\beta)) \leq 1$ in $C(S)$ in this subcase.

Now we suppose both endpoints of $\beta$ lie in $\Gamma_\alpha$. Note that if $\Gamma_\alpha$ is a single boundary circle of $S$, then a single boundary circle of $S$ contains all 4 endpoints of $\alpha$ and $\beta$. As $S$ is a 4-hole sphere and $\alpha$ is not parallel to $\beta$, there is only one configuration for $\alpha$ and $\beta$ in $S$, as shown in Figure 2.1(a), and $S - N(\alpha \cup \beta)$ consists of 3 annuli. It follows from the definition of $\pi_0$ that in this subcase, $\pi_0(\alpha)$ intersects $\pi_0(\beta)$ in two points. Hence, in this subcase, $d(\pi_0(\alpha), \pi_0(\beta)) = 1$ in $C(S)$.

![Figure 2.1](image)

**Figure 2.1.**

So we may assume next that $\Gamma_\alpha$ consists of two boundary circles of $S$ and $\partial \beta \subset \Gamma_\alpha$. By the definition of $\Gamma_\alpha$, this means that each component of $\Gamma_\alpha$ contains exactly one endpoint of $\alpha$. Although it is possible that both endpoints of $\beta$ lie in the same circle in $\Gamma_\alpha$, we can always find a component $\gamma_0$ of $\Gamma_\alpha$ such that $\gamma_0$ contains at most one endpoint of $\beta$. Since $\gamma_0$ contains exactly one endpoint of $\alpha$ and at most one endpoint of $\beta$, as illustrated in Figure 2.1(b), in any configuration of $\alpha$ and $\beta$, we can find an arc $\delta$ disjoint from $\alpha \cup \beta$ and connecting $\gamma_0$ to a component $\gamma_1$ of $\partial S - \Gamma_\alpha$. Next we consider $\pi_0(\delta)$. Since $\delta \cup \gamma_1$ is disjoint from $\alpha \cup \beta$ and by our assumption on $\gamma_0$, $\pi_0(\delta) \cap \alpha$ contains exactly one point and $\pi_0(\delta) \cap \beta$ contains at most one point. This implies that $\pi_0(\delta) \cap \pi_0(\alpha)$ contains exactly two points and $\pi_0(\delta) \cap \pi_0(\beta)$ contains at most two points. Hence $d(\pi_0(\delta), \pi_0(\alpha)) = 1$ and
Suppose an arc $\gamma$ in $\partial D$ contains a closed curve. Let $c$ be a closed curve in $D \cap E$ that is innermost in $D$. So $c$ bounds a subdisk $D_c$ of $D$ such that $D_c \cap E = c$. The closed curve $c$ also bounds a subdisk $E_c$ of $E$. As $D_c \cap E = c$, $D_c \cup E_c$ is an embedded 2–sphere in $M$. Since $M$ is irreducible, $D_c \cup E_c$
bounds a 3–ball in \( M \). We can perform an isotopy on \( E \) to eliminate the double curve \( c \) by pushing \( E_c \) across this 3–ball. This isotopy is essentially replacing \( E \) by \( (E - E_c) \cup D_c \). After a small perturbation, \( (E - E_c) \cup D_c \) has fewer intersection curves with \( D \). Thus after finitely many such isotopies, \( D \cap E \) contains no closed curve and part (1) of the lemma holds. Moreover, part (1) implies that after these isotopies, we can assume \(|D \cap E| \) is minimal in the isotopy classes of \( D \) and \( E \).

Now we prove part (2). Let \( E_{\gamma} \) be the subdisk of \( E \) cut off by \( \gamma \) as in the hypothesis of part(2). Suppose the arc \( \gamma \) is also edge-parallel in \( D \). Then \( \gamma \) and a subarc of the edge of \( \partial D \) that contains \( \partial \gamma \) bound a subdisk \( D_\gamma \) of \( D \). Since \( E_\gamma \cap D = \gamma \), \( E_\gamma \cup D_\gamma \) is a disk properly embedded in \( M \). As \( \gamma \) is edge-parallel in both \( D \) and \( E \), the construction of \( D_\gamma \) and \( E_\gamma \) implies that the boundary of the disk \( E_\gamma \cup D_\gamma \) lies in \( F - \partial S \). Since \( F - \partial S \) is incompressible in \( M \), \( \partial (E_\gamma \cup D_\gamma) \) is a trivial curve in \( F - \partial S \) and the disk \( E_\gamma \cup D_\gamma \) is \( \partial \)-parallel in \( M \). We can perform an isotopy on \( D \) to eliminate the double curve \( \gamma \) by pushing \( D_\gamma \) across \( E_\gamma \) (this isotopy basically changes \( D \) into the disk \( (D - D_\gamma) \cup E_\gamma \)). After a small perturbation, the disk after this isotopy has fewer intersection arcs with \( E \) than \(|D \cap E| \). This contradicts the assumption that \(|D \cap E| \) is minimal. \( \square \)

**Remark 3.2.** Part (1) of Lemma 3.1 implies that for any pair of compressing disks \( D \) and \( E \) in \( M \), we can assume \(|D \cap E| \), \(|\partial D \cap \partial S| \) and \(|\partial E \cap \partial S| \) are minimal in their isotopy classes at the same time.

From now on we will use \( N \) to denote the minimal value of \(|\partial D \cap \partial S| \) among all the compressing disks \( D \) in \( M \). As \( \partial S \) is separating in \( F \), \( N \) is an even number.

**Lemma 3.3.** Let \( N \) be as above and let \( D \) be a compressing disk realizing \( N \), i.e. \( D \) is an \( N \)-gon. Let \( E \) be any other compressing disk in \( M \). Suppose \(|D \cap E| \) is minimal in the isotopy classes of \( D \) and \( E \).

1. Let \( c \) be an arc properly embedded in \( D \) with endpoints in different \( \alpha \)-edges (resp. \( \beta \)-edges) of \( \partial D \), then \( c \) is not parallel in \( M \) to any arc in \( S \) (resp. \( F - \text{int}(S) \)) connecting the two endpoints of \( c \).
2. No arc of \( D \cap E \) is edge-parallel in \( E \).

**Proof.** We first prove part (1). Without loss of generality, we may suppose the two endpoints of \( c \) lie in different \( \alpha \)-edges of \( \partial D \) and the case that they lie in different \( \beta \)-edges is the same.

Suppose part (1) is false and there is an arc \( c' \) in \( S \) with \( \partial c' = \partial c \) and \( c \cup c' \) bounding an embedded disk \( \Delta \). We may suppose \( \Delta \) is transverse to \( D \) and assume \(|D \cap \Delta| \) is minimal among all such \( \Delta \). Note that as \( c' \subset S \), the endpoints of the arcs in \( D \cap \Delta \) all lie in the \( \alpha \)-edges of \( \partial D \).

If \( D \cap \Delta \) contains a closed curve, then a closed curve \( \delta \) in \( D \cap \Delta \) that is innermost in \( D \) bounds two disks \( d_1 \) and \( d_2 \) in \( D \) and \( \Delta \) respectively. As \( \delta \) is innermost in \( D \), \( d_1 \cup d_2 \) is an embedded 2–sphere. Since \( M \) is irreducible, \( d_1 \cup d_2 \) bounds a 3–ball. Similar to the proof of part (1) of Lemma 3.1
by pushing \( d_2 \) across this 3–ball, we can eliminate the double curve \( \delta \) and reduce \( |D \cap \Delta| \), a contradiction to our assumption that \( |D \cap \Delta| \) is minimal in their isotopy classes. Thus \( D \cap \Delta \) contains no closed curve.

If an arc in \( D \cap \Delta \) has both endpoints in the same \( \alpha \)–edge of \( \partial D \), then we can find such an arc \( \gamma \) of \( D \cap \Delta \) that is outermost in \( D \), i.e. \( \partial \gamma \) lies in the same \( \alpha \)–edge of \( \partial D \) and the union of \( \gamma \) and a subarc of this \( \alpha \)–edge bound a subdisk \( d_\gamma \) of \( D \) with \( d_\gamma \cap \Delta = \gamma \). Now we view \( \gamma \) as an arc in \( \Delta \). Clearly \( \partial \gamma \subset c' \), and the union of \( \gamma \) and a subarc of \( c' \) bound a subdisk \( d'_\gamma \) in \( \Delta \). As \( d'_\gamma \cap \Delta = \gamma \), \( d_\gamma \cup d'_\gamma \) is a disk properly embedded in \( M \) with boundary in \( S \). Since \( S \) is incompressible in \( M \), \( \partial(d_\gamma \cup d'_\gamma) \) bounds a disk in \( S \) and \( d_\gamma \cup d'_\gamma \) is \( \partial \)–parallel in \( M \). Similar to the proof of part (2) of Lemma 3.1, by pushing \( d'_\gamma \) across \( d_\gamma \), we can isotope \( \Delta \) to eliminate the double arc \( \gamma \) and reduce \( |D \cap \Delta| \), a contradiction to our assumption that \( |D \cap \Delta| \) is minimal. Thus every arc in \( D \cap \Delta \) has endpoints in different \( \alpha \)–edges of \( \partial D \).

Let \( \eta \) be an arc in \( D \cap \Delta \) that is outermost in \( \Delta \), i.e., \( \eta \) and a subarc of \( c' \) bound a subdisk \( \Delta' \) of \( \Delta \) with \( \Delta' \cap D = \eta \). Note that if \( D \cap \Delta = c \) then \( \eta = c \) and \( \Delta' = \Delta \). Now we view \( \eta \) as an arc in \( D \). By our conclusion above, the two endpoints of \( \eta \) lie in different \( \alpha \)–edges of \( \partial D \). The arc \( \eta \) divides \( D \) into two subdisks \( D_1 \) and \( D_2 \). As \( \Delta' \cap D = \eta \), \( D_i \cup \Delta' \) is a disk properly embedded in \( M \) for both \( i = 1, 2 \). The two disks \( D_1 \cup \Delta' \) and \( D_2 \cup \Delta' \) can be viewed as the pair of disks obtained by a \( \partial \)–compression on \( D \) along the disk \( \Delta' \). Since \( D \) is an essential disk, \( D_1 \cup \Delta' \) and \( D_2 \cup \Delta' \) cannot both be \( \partial \)–parallel in \( M \). So at least one of the two disks, say \( D_1 \cup \Delta' \), is a compressing disk in \( M \). Since the two endpoints of \( \eta \) lie in different \( \alpha \)–edges of \( \partial D \) but \( c' \cap \partial S = \emptyset \), \( D_1 \cup \Delta' \) has fewer \( \alpha \)–edges than \( D \), which contradicts that \( N = |\partial D \cap \partial S| \) is minimal among all compressing disks \( D \) of \( M \). Thus part (1) of the lemma holds.

Now we consider \( D \cap E \). Since \( |D \cap E| \) is minimal in their isotopy classes, by part (1) of Lemma 3.1, \( D \cap E \) does not contain any closed curve. Suppose part (2) of the lemma is false and there is a component of \( D \cap E \) with both endpoints in the same edge \( e \) of \( \partial E \). Let \( p \) be an outermost such arc, i.e., (1) \( \partial p \subset e \) and (2) \( p \) and a subarc of \( e \) bound a subdisk \( \Delta_p \) in \( E \) with \( \Delta_p \cap D = p \).

Since \( \partial p \subset e \), the endpoints of \( p \) either both lie in \( \alpha \)–edges of \( \partial D \) or both lie in \( \beta \)–edges of \( D \). By part (2) of Lemma 3.1, \( p \) is not edge-parallel in \( D \). Hence the two endpoints of \( p \) lie in different edges of \( \partial D \). However, as \( \Delta_p \cap D = p \), \( p \) is parallel in \( M \) to the subarc of \( e \) bounded by \( \partial p \) and this contradicts part (1) of the lemma.

**Lemma 3.4.** Let \( N \) be the minimal value of \( |\partial D \cap \partial S| \) among all the compressing disks \( D \) of \( M \). Then either

(1) the image \( \pi_A(D) \) of the disk complex lies in a ball of radius 3 in \( \mathcal{AC}(S) \), in particular, \( \pi_A(D) \) has diameter at most 6 in \( \mathcal{AC}(S) \), or
(2) \( N \) must be either 2 or 4. In other words, there must be a compressing disk that is either a bigon or a quadrilateral.
Proof. By our hypotheses, \(N\) must be an even number. Suppose part (2) is false and \(N > 4\). Let \(D\) be an \(N\)-gon. If \(d(\pi_A(\partial D), \pi_A(u)) \leq 3\) in \(\mathcal{AC}(S)\) for every \(u \in D\), then \(\pi_A(D)\) lies in the ball centered at \(\pi_A(\partial D)\) with radius 3 in \(\mathcal{AS}(S)\) and part (1) of the lemma holds.

Suppose part (1) of the lemma is also false. Then there must be a compressing disk \(E\) with \(d(\pi_A(\partial D), \pi_A(\partial E)) \geq 4\) in \(\mathcal{AC}(S)\). We may assume \(|D \cap \partial S|, |E \cap \partial S|\) and \(|D \cap E|\) are minimal in the isotopy classes of \(D\) and \(E\).

We claim that every \(\alpha\)-edge of \(\partial D\) must intersect every \(\alpha\)-edge of \(\partial E\). Suppose the claim is false, then there is an \(\alpha\)-edge \(\alpha_D\) of \(\partial D\) and an \(\alpha\)-edge \(\alpha_E\) of \(\partial E\) such that \(\alpha_D \cap \alpha_E = \emptyset\). Note that \(\pi_A(\partial D)\) and \(\pi_A(\partial E)\) are \(\alpha\)-edges of \(\partial D\) and \(\partial E\) respectively. So \(d(\alpha_D, \pi_A(\partial D)) \leq 1\) and \(d(\alpha_E, \pi_A(\partial E)) \leq 1\).

As \(\alpha_D \cap \alpha_E = \emptyset\), we have \(d(\alpha_D, \alpha_E) \leq 1\). Thus \(d(\pi_A(\partial D), \pi_A(\partial E)) \leq d(\pi_A(\partial D), \alpha_D) + d(\alpha_D, \alpha_E) + (\alpha_E, \pi_A(\partial E)) \leq 1 + 1 + 1 = 3\). This contradicts our assumption that \(d(\pi_A(\partial D), \pi_A(\partial E)) \geq 4\). Therefore each \(\alpha\)-edge in \(\partial D\) intersects every \(\alpha\)-edge in \(\partial E\).

Next we consider \(D \cap E\). First, by part (1) of Lemma 3.1, \(D \cap E\) contains no closed curve. Let \(\delta\) be an arc in \(D \cap E\) that is outermost in \(E\), i.e., there is an arc \(\delta'\) in \(\partial E\) such that \(\partial \delta' = \partial \delta\) and the subdisk \(\Delta\) of \(E\) bounded by \(\delta \cup \delta'\) has the property that \(\Delta \cap D = \delta\). In particular, \(\delta' \cap D = \partial \delta'\).

By part (2) of Lemma 3.3, \(\delta\) is not edge-parallel in \(E\) and hence \(\delta'\) contains at least one vertex (i.e., at least one point of \(\partial E \cap \partial S\)). Since each \(\alpha\)-edge of \(\partial E\) intersects every \(\alpha\)-edge of \(\partial D\) and since \(\delta' \cap D = \partial \delta'\), \(\delta'\) does not contain a whole \(\alpha\)-edge. Hence, \(\delta' \cap \partial S\) either is a single point, in which case \(\Delta\) is a triangle (see Figure 3.1(a)), or contains exactly two points, in which case \(\delta'\) contains the whole of a \(\beta\)-edge and \(\Delta\) is a quadrilateral (see Figure 3.1(b)).

![Figure 3.1.](image)

Case (i). \(\Delta\) is a triangle, i.e., \(\partial \Delta\) consists of \(\delta\), a subarc of an \(\alpha\)-edge and a subarc of a \(\beta\)-edge.

In this case, one endpoint of \(\delta\) lies in an \(\alpha\)-edge and the other endpoint of \(\delta\) lies in a \(\beta\)-edge of \(\partial D\). Hence, both components of \(\partial D - \partial \delta\) intersect \(\partial S\).
The arc $\delta$ divides $D$ into two subdisks which we denote by $D_1$ and $D_2$. If $D_1$ is a triangle (i.e., $\partial D_1 \cap \partial S$ is a single point), then $D_1 \cup \Delta$ is a bigon disk properly embedded in $M$ which intersects $S$ twice. Since we have assumed $N > 4$, $D_1 \cup \Delta$ must be a $\partial$–parallel disk in $M$. So $D_2 \cup \Delta$ is isotopic to $D$. As $\Delta \cap D = \delta$, after perturbing $D_2 \cup \Delta$ a little, we get a disk isotopic to $D$ but having fewer intersection arcs with $E$ than $|D \cap E|$, a contradiction to our assumption that $|D \cap E|$ is minimal in their isotopy classes. Thus, we may assume that $\partial D_i \cap \partial S$ contains more than one point for both $i = 1, 2$. Therefore, both $D_1 \cup \Delta$ and $D_2 \cup \Delta$ have fewer intersection points with $\partial S$ than $|\partial D \cap \partial S|$. Moreover, similar to the proof of Lemma 3.3, $D_1 \cup \Delta$ and $D_2 \cup \Delta$ can be viewed as the two disks obtained by $\partial$–compressing $D$ along $\Delta$. Hence at least one of $D_1 \cup \Delta$ and $D_2 \cup \Delta$ is not $\partial$–parallel in $M$. As both disks $D_1 \cup \Delta$ and $D_2 \cup \Delta$ have fewer intersection points with $\partial S$ than $|\partial D \cap \partial S|$, this contradicts the assumption that $N = |\partial D \cap \partial S|$ is minimal among all compressing disks $D$ of $M$.

Case (ii). $\Delta$ is a quadrilateral (i.e., $\delta'$ contains a whole $\beta$–edge of $\partial E$ and $|\delta' \cap \partial S| = 2$).

This case is similar. As $\delta'$ contains a whole $\beta$–edge of $\partial E$ and $|\delta' \cap \partial S| = 2$, both endpoints of $\delta$ lie in $\alpha$–edges. The arc $\delta$ divides $D$ into two subdisks $D_1$ and $D_2$. As both endpoints of $\delta$ lie in $\alpha$–edges, both $|\partial D_1 \cap \partial S|$ and $|\partial D_2 \cap \partial S|$ are even numbers.

If $\partial D_1 \cap \partial S = \emptyset$ (i.e., $\partial \delta$ lies in the same edge of $\partial D$), then $D_1 \cup \Delta$ is a bigon disk properly embedded in $M$. Since we have assumed $N > 4$, $D_1 \cup \Delta$ must be $\partial$–parallel in $M$ and the closed curve $\partial(D_1 \cup \Delta)$ bounds a disk $\Delta_M$ in $\partial M$. As $D_1 \cup \Delta$ is a bigon disk, $\partial S \cap \partial \Delta_M$ consists of 2 points. Hence $\partial S \cap \Delta_M$ is a single arc properly embedded in $\Delta_M$ connecting the two points in $\partial S \cap \partial \Delta_M$. By our assumption on $\Delta$ in this case, $\partial \Delta$ contains a whole $\beta$–edge $\beta_E$ of $\partial E$. Moreover, this $\beta$–edge $\beta_E$ is an arc in $\partial \Delta_M$ ($\partial \Delta_M = \partial(D_1 \cup \Delta)$) and $|\partial \Delta_M \cap \partial S| = \emptyset$. This means that $\beta_E$ is parallel to the arc $\partial S \cap \Delta_M$ and hence $\beta_E$ is $\partial$–parallel in $F - int(S)$. This contradicts our assumption at the beginning that $|\partial E \cap \partial S|$ is minimal and in particular, each $\beta$–edge of $\partial E$ is an essential arc in $F - int(S)$. So we may assume that $\partial D_i \cap \partial S \neq \emptyset$ for both $i = 1, 2$.

If $\partial D_1 \cap \partial S$ contains exactly two points, then since $\partial \delta$ lies in $\alpha$–edges, $\partial D_1$ contains the whole of a $\beta$–edge of $\partial D$, and $D_1 \cup \Delta$ is a quadrilateral disk properly embedded in $M$. Again, since $N > 4$, $D_1 \cup \Delta$ must be $\partial$–parallel in $M$. Thus, $D_2 \cup \Delta$ is isotopic to $D$. As $\Delta \cap D = \delta$, after perturbing $D_2 \cup \Delta$ a little we get a disk isotopic to $D$ but having fewer intersection arcs with $E$ than $|D \cap E|$, a contradiction.

So we may assume $D_1 \cap \partial S$ contains more than two points for both $i = 1, 2$. This implies that, for both $i = 1, 2$, $D_i \cup \Delta$ is a disk whose intersection with $\partial S$ contains fewer points than $|\partial D \cap \partial S| = N$. Similar to Case (i) above, we may view $D_1 \cup \Delta$ and $D_2 \cup \Delta$ as the pair of disks obtained by $\partial$–compressing $D$ along $\Delta$. Since $D$ is an essential disk in $M$, at least one of $D_1 \cup \Delta$ and
$D_2 \cup \Delta$ is not $\partial$–parallel in $M$. As both disks $D_1 \cup \Delta$ and $D_2 \cup \Delta$ have few intersection points with $\partial S$ than $|\partial D \cap \partial S|$, this contradicts the assumption that $N = |\partial D \cap \partial S|$ is minimal among all compressing disks $D$ of $M$. \hfill \Box

Lemma 3.3 reduces the proof of Theorem 3 into two cases: $N = 2$ and $N = 4$.

4. Case 1: $N = 2$

In this section, we prove Theorem 3 in the case that $N = 2$, that is, $M$ contains at least one bigon compressing disk.

As in section 1 for any $I$–bundle $J$ over a compact surface, we use $\partial_h J$ and $\partial_v J$ to denote the horizontal and vertical boundary of $J$ respectively. In particular, $\partial_v J$ is a collection of annuli.

**Definition 4.1.** Let $J$ be a compact submanifold of $M$ and let $Q$ be a subsurface of $F \subset \partial M$. We call $J$ an $I$–bundle region in $M$ with respect to $Q$ if

1. $J$ is an $I$–bundle over a compact surface $P$ with $\partial P \neq \emptyset$,
2. the horizontal boundary $\partial_h J$ is a subsurface of $Q$, and $J \cap Q = \partial_h J$,
3. $\partial_v J \cap (F - Q) \neq \emptyset$ and each component of $\partial_v J \cap (F - Q)$ is either an annular component of $\partial_v J$ or a rectangular disk $I \times I$, where $I \times \partial I \subset \partial Q$ and $\partial I \times I$ is a pair of properly embedded essential arcs in both $\partial_v J$ and $F - \text{int}(Q)$.

Any bigon $D$ in $M$ can be viewed as a product disk as follows. Let $A$ be a small neighborhood of $\partial S$ in $F - \text{int}(S)$, and we denote the closure of $F - (S \cup A)$ by $S'$. Then we can view $D$ as a product $I \times I$ with $\partial I \times I$ a pair of vertical arcs in $A$ and with $I \times \partial I$ a pair of arcs in $S$ and $S'$. Moreover, a small neighborhood of $D$ can be viewed as an $I$–bundle region with respect to $S \cup S'$, see Definition 4.1.

Since $F - \partial S$ is incompressible in $M$, as in [3] (also see [1] Lemma 2.1), $M$ has a maximal essential $I$–bundle region $J$ (with respect to $S \cup S'$ above) such that any bigon disk (or product disk described above) can be isotoped into $J$. We refer the reader to [1] Lemma 2.1 for a simple proof of the existence of $J$ (in fact the argument in the next section of this paper can also prove the existence of $J$). Since $\partial S$ is separating in $F$ and $\partial J \cap A \neq \emptyset$, $J$ is a product of a compact surface and an interval $I$. Note that $J \cap S$ is a collection of components of the horizontal boundary of $J$. Moreover, as $J$ is maximal, after some isotopy on $J$ if necessary, either $J \cap S = S$ or $J \cap S$ is a non-trivial subsurface of $S$ whose frontier in $S$ consists of essential arcs and essential non-peripheral closed curves in $S$.

If $J \cap S = S$, then $J$ is a product $S \times I$ and part (1) of Theorem 3 holds. So we may suppose next that $J \cap S$ is a non-trivial subsurface of $S$.

Let $\alpha$ be any arc or closed curve in the frontier of $J \cap S$ in $S$. By our assumption on $J$ above, $\alpha$ is essential and non-peripheral in $S$, representing a vertex of $\mathcal{AC}(S)$.
If \( d(\alpha, \pi_A(\partial E)) \leq 3 \) for every compressing disk \( E \), then \( \pi_A(D) \) lies in the 3–ball centered at \( \alpha \) with radius 3 in \( \mathcal{A}(S) \) and Theorem 3.1 holds. Suppose Theorem 3 is false and let \( E \) be a compressing disk with \( d(\alpha, \pi_A(\partial E)) \geq 4 \).

Next we fix \( E \) and let \( D \) be a bigon compressing disk such that \(|D \cap E|\) is minimal among all the bigon compressing disks. Since \( D \) is a bigon, the \( \alpha \)–edge of \( \partial D \) is the arc \( D \cap S \) and \( \pi_A(\partial D) = D \cap S \). Since \( D \) can be isotoped into \( J \), the distance between \( \alpha \) and \( D \cap S \) in \( \mathcal{A}(S) \) is at most one, i.e. \( d(\alpha, \pi_A(\partial D)) \leq 1 \).

If \( E \) has an \( \alpha \)–edge \( \alpha_E \) disjoint from \( D \), then \( d(\pi_A(\partial D), \alpha_E) \leq 1 \) and \( d(\pi_A(\partial D), \pi_A(\partial E)) \leq d(\pi_A(\partial D), \alpha_E) + d(\alpha_E, \pi_A(\partial E)) \leq 1 + 1 = 2 \). Hence we have \( d(\alpha, \pi_A(\partial E)) \leq d(\alpha, \pi_A(\partial D)) + d(\pi_A(\partial D), \pi_A(\partial E)) \leq 1 + 2 = 3 \), which contradicts our assumption on \( E \) that \( d(\alpha, \pi_A(\partial E)) \geq 4 \). So every \( \alpha \)–edge of \( \partial E \) intersects \( \partial D \).

Next we consider \( D \cap E \). As before, we view \( E \) as a polygon with \( E \cap \partial S \) being its vertices, and we may assume \(|E \cap \partial S|\) and \(|E \cap D|\) are minimal in their isotopy classes. By part (1) of Lemma 3.1 \( E \cap D \) contains no closed curve.

Let \( \delta \) be a component of \( D \cap E \) that is outermost in \( E \). So there is a subarc \( \delta' \) of \( \partial E \) with \( \partial \delta' = \partial \delta \) and such that \( \delta \cup \delta' \) bounds a subdisk \( \Delta \) of \( E \) with \( \Delta \cap D = \delta \). By part (2) of Lemma 3.3, the two endpoints of \( \delta \) lie in different edges of \( E \). Hence \( \delta' \cap \partial S \neq \emptyset \). Since \( \partial D \) intersects every \( \alpha \)–edge of \( \partial E \), as in the proof of Lemma 3.4, \( \delta' \) does not contain a whole \( \alpha \)–edge and \( \Delta \) is either a triangle (in which case \( \delta' \cap \partial S \) is a single point) or a quadrilateral (in which case \( \delta' \cap \partial S \) consists of two points and \( \delta' \) contains the whole of a \( \beta \)–edge).

We first suppose \( \Delta \) is a triangle, i.e. one endpoint of \( \delta \) lies in an \( \alpha \)–edge and the other endpoint of \( \delta \) lies in a \( \beta \)–edge. The arc \( \delta \) cuts \( D \) into two subdisks \( D_1 \) and \( D_2 \). As \( D \) is a bigon disk, \( \delta \) must be an arc in \( D \) which separates the two vertices of \( \partial D \), and both \( D_1 \cup \Delta \) and \( D_2 \cup \Delta \) are bigon disks properly embedded in \( M \). Since \( D \) is a compressing disk and \( \Delta \cap D = \delta \), \( D_1 \cup \Delta \) and \( D_2 \cup \Delta \) cannot both be \( \partial \)–parallel in \( M \). After some small perturbation, both \( D_1 \cup \Delta \) and \( D_2 \cup \Delta \) have fewer intersection arcs with \( E \) than \(|D \cap E|\), which contradicts the assumption that \(|D \cap E|\) is minimal among all bigon compressing disks \( D \). Therefore, \( \Delta \) must be a quadrilateral.

In the case that \( \Delta \) is a quadrilateral, \( \delta' \) contains a whole \( \beta \)–edge of \( \partial E \) and hence both endpoints of \( \delta \) lie in the \( \alpha \)–edges of \( \partial E \). Hence \( \partial \delta \) lies in the \( \alpha \)–edge of \( \partial D \) and \( \delta \) is an edge-parallel arc in the bigon \( D \). Let \( D_1 \) be the subdisk of \( D \) bounded by \( \delta \) and a subarc of the \( \alpha \)–edge \( D \cap S \). Then \( D_1 \cup \Delta \) is a bigon disk properly embedded in \( M \). If \( D_1 \cup \Delta \) is a \( \partial \)–parallel disk in \( M \), then similar to Case (ii) in the proof of Lemma 3.4, the \( \beta \)–edge of \( \partial E \) that lies in \( \delta' \) must be a \( \partial \)–parallel arc in \( F - \text{int}(S) \), which contradicts our assumption at the beginning that \( |\partial E \cap \partial S| \) is minimal in the isotopy class of \( E \) (in particular each \( \beta \)–edge of \( \partial E \) is essential in \( F - \text{int}(S) \)). Thus the bigon \( D_1 \cup \Delta \) is a compressing disk in \( M \). However, after perturbing \( D_1 \cup \Delta \) a little, \((D_1 \cup \Delta) \cap E \) has fewer components than \(|D \cap E|\), which
again contradicts the assumption that $|D \cap E|$ is minimal among all bigon compressing disks $D$.

Therefore, Theorem 3 holds in the case that $N = 2$.

5. CASE 2: $N = 4$

In this section, we prove Theorem 3 in the case that $N = 4$, that is, $M$ contains at least one quadrilateral compressing disk but $M$ contains no bigon compressing disk.

The proof is similar to the case $N = 2$. We first build a maximal $I$–bundle region with respect to $S$. Note that unlike the case $N = 2$, in which case the $I$–bundle region is with respect to $S \cup S'$, we cannot directly quote the results on characteristic submanifolds in [3], as $F - (S \cup S')$ is a collection of annuli in section 4 while $F - S$ is not. In this section, all the $I$–bundle regions are with respect to $S$.

**Definition 5.1.** Let $J$ be an $I$–bundle region in $M$ with respect to $S$, see Definition 4.1. We say that $J$ is an essential $I$–bundle region if the frontier of $\partial hJ$ in $S$ consists of non-trivial arcs and closed curves in $S$.

Note that given any quadrilateral compressing disk $D$ of $M$, a small neighborhood of $D$ in $M$ is an essential $I$–bundle region with respect to $S$. Moreover, for any quadrilateral compressing disk $D$ of $M$ that lies in an essential $I$–bundle region $J$, one can isotope $D$ to be vertical in $J$, i.e. $D$ is the union of a collection of $I$–fibers of $J$ after isotopy.

**Lemma 5.2.** Let $J$ be an $I$–bundle region in $M$ with respect to $S$ as in Definition 4.1. Then $J$ can be extended to an essential $I$–bundle region in $M$ with respect to $S$.

**Proof.** Let $V$ be the frontier of $J$ in $M$. By our definition of $J$, $V \subset \partial vJ$ and $V$ consists of annuli and quadrilaterial disks properly embedded in $M$. Moreover, $V \cap S$ is the frontier of $\partial hJ$ in $S$.

If a simple closed curve $\gamma_0$ in the frontier of $\partial hJ$ in $S$ is trivial in $S$, then $\gamma_0$ bounds a disk $D_0$ in $S$. Since each component of $J$ contains a quadrilateral compressing disk, $D_0 \cap \partial hJ = \gamma_0$. Let $\Gamma$ be the component of $V$ that contains $\gamma_0$. So $\Gamma$ is an annulus and $\gamma_0$ is a component of $\partial \Gamma$. Let $\gamma_1$ be the other component of $\partial \Gamma$. Since $\gamma_1$ bounds a disk $\Gamma \cup D_0$ and since $F - \partial S$ is incompressible, $\gamma_1$ must bound a disk $D_1$ in $S$. Moreover, $D_0 \cup \Gamma \cup D_1$ is an embedded 2–sphere in $M$ bounding a 3–ball. We can give the 3–ball bounded by $D_0 \cup \Gamma \cup D_1$ a product structure $D^2 \times I$ with $D^2 \times \partial I = D_0 \cup D_1$ and $\partial D^2 \times I = \Gamma$. By adding this 3–ball (with this product structure) to $J$, we obtain a larger $I$–bundle region.

If an arc $\alpha_0$ in the frontier of $\partial hJ$ in $S$ is a $\partial$–parallel arc in $S$, then $\alpha_0$ and a subarc of $\partial S$ bound a subdisk $\Delta_0$ of $S$. Since each component of $J$ contains a quadrilateral compressing disk, $\Delta_0 \cap \partial hJ = \alpha_0$. Let $Q$ be the component of $V$ containing $\alpha_0$. So $Q$ quadrilateral disk in $M$ and $\alpha_0$ can be viewed as an $\alpha$–edge of $Q$. If $Q$ is a compressing disk in $M$, since $\alpha_0$ is a trivial arc
in \( S \), one can isotope \( \partial Q \) pushing \( \alpha_0 \) across the disk \( \Delta_0 \) and this isotopy reduces \(|\partial Q \cap \partial S|\) from 4 to 2. This contradicts the hypothesis that \( N = 4 \) in this case. So \( Q \) must be a \( \partial \)-parallel disk in \( M \) and \( \partial Q \) bounds a disk \( \Delta_Q \) in \( \partial M \). As \(|\partial Q \cap \partial S| = 4\), \( \partial S \cap \Delta_Q \) is a pair of arcs properly embedded in \( \Delta_Q \), one of which is \( \alpha_0 \). We can give the 3–ball bounded by \( Q \cup \Delta_Q \) a product structure \( \Delta \times I \) where \( \Delta_0 \times \{0\} = \Delta_0 \) and \( Q = \alpha_0 \times I \subset \partial \Delta_0 \times I \). By adding this 3–ball (with this product structure) to \( J \), we obtain a larger \( I \)-bundle region.

Thus after adding such 3–balls to \( J \), we get a larger \( I \)-bundle region \( \hat{J} \) such that the frontier of \( \partial \hat{J} \) in \( S \) consists of non-trivial arcs and closed curves in \( S \). By definition, \( \hat{J} \) is an essential \( I \)-bundle region in \( M \). □

**Definition 5.3.** We say that an essential \( I \)-bundle region \( J \) is maximal if for any essential \( I \)-bundle region \( J' \) containing \( J \), then \( J' \) and \( \partial \hat{J} \) are isotopic to \( J \) and \( \partial \hat{J} \) in \( M \) and \( S \) respectively.

For any essential \( I \)-bundle region \( J \) in \( M \) with respect to \( S \), since \( \partial \hat{J} \subset S \), if \( \partial \hat{J} \) is isotopic to \( S \) in \( F \), then part (1) of Theorem \[\text{B}\] holds. Suppose part (1) of Theorem \[\text{B}\] is false, so we may assume next that a component of the frontier of \( \partial \hat{J} \) in \( S \) is either an essential arc or an essential non-peripheral closed curve in \( S \).

**Lemma 5.4.** Let \( J \) be a maximal essential \( I \)-bundle as above. Let \( D_1 \) be any quadrilateral compressing disk in \( M \). Then each \( \alpha \)-edge of \( \partial D_1 \) can be isotoped either totally into \( \partial \hat{J} \) or totally out of \( \partial \hat{J} \), in other words, each \( \alpha \)-edge of \( \partial D_1 \) can be isotoped to be disjoint from the frontier of \( \partial \hat{J} \) in \( S \).

**Proof.** Let \( V \) be the frontier of \( J \) in \( M \). Since \( J \) is an essential \( I \)-bundle and since \( F - \partial S \) is incompressible in \( M \), a component of \( V \) is either an incompressible annulus with boundary in \( S \) or a quadrilateral compressing disk for \( M \). Let \( \Gamma \) be the frontier of \( \partial \hat{J} \) in \( S \). So \( \Gamma = V \cap S \) and \( \Gamma \) is the union of the boundary curves of the annular components of \( V \) and the \( \alpha \)-edges of the quadrilateral components of \( V \). Thus if every \( \alpha \)-edge of \( D_1 \) is disjoint from \( \Gamma \), then the lemma holds. Suppose the lemma is false and an \( \alpha \)-edge of \( D_1 \) non-trivially intersects \( \Gamma \).

Now we consider \( D_1 \cap V \). We may suppose \(|D_1 \cap V|\) is minimal up to isotopy on \( D_1 \).

If \( D_1 \cap V \) contains a closed curve, then since each annular component of \( V \) is incompressible in \( M \), the closed curve must bound disks in both \( D_1 \) and \( V \). Similar to part (1) of Lemma \[\text{3.4}\], we can isotope \( D_1 \) to eliminate a closed intersection curve and reduce \(|D_1 \cap V|\). As we have assumed \(|D_1 \cap V|\) is minimal, \( D_1 \cap V \) contains no closed curve.

We say a properly embedded arc \( \gamma \) in \( V \) is edge-parallel in \( V \) if either \( \gamma \) is edge-parallel in a quadrilateral component of \( V \) as before, or \( \gamma \) is \( \partial \)-parallel in an annular component of \( V \).

**Claim 1.** No arc of \( D_1 \cap V \) is edge-parallel in \( V \).
Proof of Claim 1. Suppose the claim is false. Then we can find an outermost edge-parallel arc $\gamma$ in $V$, i.e., $\gamma$ and a subarc of the component of $\partial V - \partial S$ containing $\partial \gamma$ bound a subdisk $E$ of $V$ with $E \cap D_1 = \gamma$. Now we view $\gamma$ as an arc in $D_1$. First, as $\gamma$ is edge-parallel in $V$, the two endpoints of $\gamma$ either both lie in $\gamma$–edges of $\partial D_1$ or both lie in $\beta$–edges of $\partial D_1$. Since $D_1$ is a quadrilateral compressing disk and since $\gamma$ is parallel in $M$ to the arc $\partial E - \text{int}(\gamma) \subset F - \partial S$, by part (1) of Lemma 3.3, the two endpoints of $\gamma$ must lie in the same edge of $\partial D_1$, i.e. $\gamma$ must be edge-parallel in $D_1$.

Now the proof is similar to part (2) of Lemma 3.3. As $\gamma$ is edge-parallel in $D_1$, $\gamma$ and a subarc of the edge of $\partial D_1$ that contains $\partial \gamma$ bound a subdisk $E'$ of $D_1$. Since $E \cap D_1 = \gamma$, $E \cup E'$ is a properly embedded disk in $M$ with boundary circle disjoint from $\partial S$. Since $F - \partial S$ is incompressible in $M$, $\partial(E \cup E')$ bounds a disk in $F - \partial S$ which together with $E \cup E'$ form a 2–sphere bounding a 3–ball. So we can perform an isotopy on $D_1$ pushing $E'$ across this 3–ball to eliminate $\gamma$ and reduce $|D_1 \cap V|$. This contradicts our assumption that $|D_1 \cap V|$ is minimal up to isotopy on $D_1$. Thus no arc in $D_1 \cap V$ is edge-parallel in $V$. \hfill \Box

Claim 2. No arc of $D_1 \cap V$ cuts off a triangle from the quadrilateral disk $D_1$. In other words, there is no arc in $D_1 \cap V$ having one endpoint in an $\alpha$–edge and the other endpoint in a $\beta$–edge of $\partial D_1$.

Proof of Claim 2. Suppose there is such an arc in $D_1 \cap V$, so this arc has one endpoint in $S$ and the other endpoint in $F - S$. Since the boundary of the annular components of $V$ lies in $S$, such an intersection arc must be in a quadrilateral component of $V$. Next we view such intersection arcs in $V$.

It follows from Claim 1 that there exists such an arc $\gamma$ of $D_1 \cap V$ that is outermost in $V$, i.e., $\gamma$ cuts off a triangle $\Delta$ in $V$ such that $\Delta \cap D_1 = \gamma$ and $\partial \Delta \cap \partial S$ is a single point. Now we view $\gamma$ as an arc in $D_1$. So one endpoint of $\gamma$ lies in an $\alpha$–edge and the other endpoint lies in a $\beta$–edge of $\partial D_1$. Since $D_1$ is a quadrilateral compressing disk, $\gamma$ also cuts off a triangle $\Delta_1$ in $D_1$ and $\Delta \cup \Delta_1$ is a properly embedded bigon disk in $M$. If $\Delta \cup \Delta_1$ is a compressing disk for $M$, then $\Delta \cup \Delta_1$ is a bigon compressing disk and $N = 2$, which contradicts our hypothesis that $N = 4$ in this section. So $\Delta \cup \Delta_1$ is a $\partial$–parallel disk in $M$. Similar to the previous case, we can perform an isotopy on $D_1$ pushing $\Delta_1$ across $\Delta$ to eliminate $\gamma$ and reduce $|D_1 \cap V|$, contradicting our assumption that $|D_1 \cap V|$ is minimal up to isotopy on $D_1$. Thus no arc of $D_1 \cap V$ cuts off a triangle from $D_1$ or $V$. \hfill \Box

Claim 3. No arc of $D_1 \cap V$ has both endpoints in the same edge of $\partial D_1$.

Proof of Claim 3. Let $\gamma$ be an outermost such arc in $D_1$, i.e., $\partial \gamma$ lies in the same edge of $\partial D_1$ and the union of $\gamma$ and a subarc of this edge bound a subdisk $E_1$ of $D_1$ with $E_1 \cap V = \gamma$. So $\gamma$ is parallel in $M$ to the arc $\partial E_1 - \text{int}(\gamma) \subset F - \partial S$.

Now we view $\gamma$ as an arc in $V$. If $\gamma$ lies in a quadrilateral component $Q$ of $V$, then by Claim 1 above, the two endpoints of $\gamma$ lie in different
α–edges (or different β–edges) of ∂Q. As γ is parallel in M to the arc ∂E1 − int(γ) ⊂ F − ∂S, this contradicts part (1) of Lemma 3.3 (since N = 4 and Q is a quadrilateral compressing disk).

If γ is an arc in an annular component of V, then by Claim 1 above, γ must be an essential arc in this annular component. So after isotopy, we may assume γ is an I–fiber of J. By the definition of J, each component of J contains a quadrilateral compressing disk of M. Let D be a quadrilateral compressing disk of M lying in the component of J that contains γ. We may isotope D to be vertical in J and so that γ ⊂ D. As γ is parallel in M to the arc ∂E1 − int(γ) ⊂ F − ∂S, this again contradicts part (1) of Lemma 3.3.

Claim 4. The two endpoints of each arc of D1 ∩ V lie in different α–edges of D1.

Proof of Claim 4. By Claim 2 and Claim 3, the two endpoints of each arc of D1 ∩ V must lie in either different α–edges or different β–edges of ∂D1.

Suppose the claim is false, then there is an arc of D1 ∩ V with two endpoints in different β–edges of ∂D1. This arc separates the two α–edges of ∂D1 in the quadrilateral disk D1. As D1 ∩ V is a collection of disjoint arcs in D1, Claim 2 and Claim 3 imply that every arc of D1 ∩ V has endpoints in different β–edges of D1. Hence the α–edges of D1 are disjoint from V, which contradicts our assumption at the beginning that an α–edge of D1 non-trivially intersects Γ = V ∩ S.

Lemma 5.4 follows from the 4 claims above. By Claim 4 and Claim 1, we may isotope D1 so that D1 ∩ V is a collection of I–fibers of J and the closure of each component of D1 − J is a rectangular disk outside J. Let J1 be the union of J and a small neighborhood of D1 in M. Clearly J1 is also an I–bundle region with respect to S. By Lemma 5.2, after adding some 3–ball components of M − J1 to J1, we can extend J1 to an essential I–bundle region J. However, since J is a maximal essential I–bundle and J ⊂ J, J must be isotopic to J. Since D1 ⊂ J, this means that we can isotope D1 into J and the lemma holds.

Now we are in position to prove Theorem 3 in the case that N = 4. Let J be a maximal essential I–bundle region with respect to S as above. Let α be a component of the frontier of ∂hJ in S. As we have assumed part (1) of Theorem 3 is false, we may assume α is either an essential arc or an essential and non-peripheral closed curve in S representing a vertex in A\mathcal{C}(S).

If d(α, πA(∂E)) ≤ 3 for every compressing disk E, then πA(D) lies in the 3–ball centered at α with radius 3 in A\mathcal{C}(S) and Theorem 3 holds. Suppose Theorem 3 is false and let E be a compressing disk with d(α, πA(∂E)) ≥ 4.

Next we fix E and let D be a quadrilateral compressing disk such that |D ∩ E| is minimal among all quadrilateral compressing disks. If an α–edge αE of ∂E is disjoint from all the α–edges of ∂D, then d(αE, πA(∂D)) ≤ 1 and d(πA(∂E), πA(∂D)) ≤ d(πA(∂E), αE) + d(αE, πA(∂D)) ≤ 1 + 1 = 2.
By Lemma 5.3, one can isotope $D$ so that $\pi_A(\partial D)$ lies either totally in $\partial bJ$ or totally outside $\partial bJ$, in particular, $\pi_A(\partial D)$ is disjoint from $\alpha$ after isotopy. So $d(\alpha, \pi_A(\partial D)) \leq 1$ and $d(\alpha, \pi_A(\partial E)) \leq d(\alpha, \pi_A(\partial D)) + d(\pi_A(\partial D), \pi_A(\partial E)) \leq 1 + 2 = 3$. This contradicts our assumption on $E$ that $d(\alpha, \pi_A(\partial E)) \geq 4$. Thus every $\alpha$-edge of $E$ must intersect $\partial D$.

By part (1) of Lemma 3.3, we may assume $D \cap E$ contains no closed curves. Let $\gamma$ be a component of $D \cap E$ that is outermost in $E$. So there is an arc $\gamma'$ in $\partial E$ such that $\partial \gamma = \partial \gamma'$ and $\gamma \cup \gamma'$ bounds a subdisk $\Delta_E$ of $E$ with $D \cap \Delta_E = \gamma$. In particular, $\gamma' \cap D = \partial \gamma'$.

By part (2) of Lemma 3.3, the two points of $\partial \gamma$ lie in different edges of $E$.

By our conclusion above that every $\alpha$-edge of $E$ intersects $\partial D$ and since $\gamma' \cap D = \partial \gamma'$, $\gamma'$ does not contain a whole $\alpha$-edge of $E$. As the two points in $\partial \gamma'$ lie in different edges of $\partial E$, the disk $\Delta_E$ is either a triangle in which $\gamma' \cap \partial S$ is a single point, or a quadrilateral in which case $|\gamma' \cap \partial S| = 2$ and $\gamma'$ contains a whole $\beta$-edge of $\partial E$. So we have the following two cases.

Case (a). $\Delta_E$ is a triangle.

In this case, one endpoint of $\gamma$ lies in an $\alpha$-edge of $\partial E$ and the other endpoint of $\gamma$ lies in a $\beta$-edge of $\partial E$. Now we view $\gamma$ as an arc in $D$. Since $D$ is a quadrilateral compressing disk, $\gamma$ also cuts off a triangular disk $\Delta_D$ in $D$. As $\Delta_E \cap \Delta_D = \gamma$, $\Delta_E \cup \Delta_D$ is a bigon disk properly embedded in $M$. Since $N = 4$ and $\Delta_E \cup \Delta_D$ is a bigon disk, $\Delta_D \cup \Delta_E$ must be $\partial$-parallel in $M$. Hence $(D - \Delta_D) \cup \Delta_E$ is a quadrilateral compressing disk isotopic to $D$. However, after a slight perturbation, $(D - \Delta_D) \cup \Delta_E$ has fewer intersection arcs with $E$ than $|D \cap E|$, contradicting our assumption that $|D \cap E|$ is minimal. So Case (a) cannot happen.

Case (b). $\Delta_E$ is a quadrilateral.

By our conclusion on $\Delta_E$ in this case, both endpoints of $\gamma$ lie in $\alpha$-edges of $\partial E$ and hence $\partial \gamma \subset S$. This implies that both points of $\partial \gamma$ lie in $\alpha$-edges of $\partial D$. We have two subcases.

The first subcase is that both points of $\partial \gamma$ lie in the same $\alpha$-edge of $\partial D$. This subcase is similar to Case (ii) in the proof of Lemma 3.3. In this subcase, $\gamma$ and a subarc of the $\alpha$-edge of $\partial D$ that contains $\partial \gamma$ bound a subdisk $\Delta_D$ of $D$. As $\Delta_E \cap \Delta_D = \gamma$, $\Delta_E \cup \Delta_D$ is a bigon disk properly embedded in $M$. As $N = 4$ and $\Delta_E \cup \Delta_D$ is a bigon disk, $\Delta_E \cup \Delta_D$ must be $\partial$-parallel in $M$ and the closed curve $\partial(\Delta_E \cup \Delta_D)$ bounds a disk $\Delta_M$ in $\partial M$. As $\Delta_E \cup \Delta_D$ is a bigon disk, $\partial S \cap \partial \Delta_M$ consists of 2 points. Hence $\partial S \cap \Delta_M$ is a single arc properly embedded in $\Delta_M$ connecting the two points in $\partial S \cap \partial \Delta_M$. Recall that the boundary of the quadrilateral $\Delta_E$ contains a whole $\beta$-edge $\beta_E$ of $\partial E$. Moreover, this $\beta$-edge $\beta_E$ is an arc in $\partial \Delta_M$ ($\partial \Delta_M = \partial(\Delta_E \cup \Delta_D)$) and $\partial \beta_E = \partial S \cap \partial \Delta_M$. This means that $\beta_E$ is parallel to the arc $\partial S \cap \Delta_M$ and hence is $\partial$-parallel in $F - \text{int}(S)$. This contradicts our assumption at the beginning that $|\partial E \cap \partial S|$ is minimal and in particular, each $\beta$-edge of $\partial E$ is an essential arc in $F - \text{int}(S)$.
The second subcase is that the two points of $\partial \gamma$ lie in different $\alpha$–edges of $\partial D$. In this subcase, $\gamma$ divides $D$ into two subdisks $D_1$ and $D_2$ and both $D_1 \cup \Delta E$ and $D_2 \cup \Delta E$ are properly embedded in $M$. Since $D$ is a quadrilateral compressing disk and since the two points of $\partial \gamma$ lie in different $\alpha$–edges of $\partial D$, both $D_1 \cup \Delta E$ and $D_2 \cup \Delta E$ are quadrilateral disks. We may view $D_1 \cup \Delta E$ and $D_2 \cup \Delta E$ as the two disks obtained by a $\partial$–compression on $D$ along $\Delta E$. As $D$ is a compressing disk, $D_1 \cup \Delta E$ and $D_2 \cup \Delta E$ cannot both be $\partial$–parallel in $M$, so at least one disk, say $D_1 \cup \Delta E$, is a quadrilateral compressing disk. However, after a slight perturbation, $D_1 \cup \Delta E$ has fewer intersection arcs with $E$ than $|D \cap E|$, contradicting our assumption that $|D \cap E|$ is minimal among all quadrilateral compressing disks $D$ in $M$.

Therefore case (b) cannot happen neither and Theorem 3 holds in the case that $N = 4$.

References


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