SMALL 3-MANIFOLDS WITH LARGE HEEGAARD DISTANCE

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ABSTRACT. We construct examples of closed non-Haken hyperbolic 3-manifolds with a Heegaard splitting of arbitrarily large distance.

1. INTRODUCTION

A Heegaard splitting of a closed orientable 3-manifold $M$ is a decomposition of $M$ into two handlebodies along an embedded surface called a Heegaard surface. A very useful tool in studying Heegaard splittings is the curve complex. Let $F$ be a closed orientable surface of genus at least 2. The curve complex of $F$, introduced by Harvey [?], is the complex whose vertices are the isotopy classes of essential simple closed curves in $F$, and $k + 1$ vertices determine a $k$-simplex if they are represented by pairwise disjoint curves. We denote the curve complex of $F$ by $C(F)$. For any two vertices $x$, $y$ in $C(F)$, the distance $d(x, y)$ is the minimal number of 1-simplices in a simplicial path jointing $x$ to $y$. If $F$ bounds a handlebody $V$, then the disk complex $D(V)$ is the subcomplex of $C(F)$ containing only the vertices represented by boundary curves of compressing disks in $V$. Given a Heegaard splitting $M = V \cup F W$, the distance $d(V, W)$, introduced by Hempel [?], is the distance between $D(V)$ and $D(W)$ in the curve complex $C(F)$.

An incompressible surface in $M$ is an embedded surface which contains no essential loop bounding a compressing disk in $M$. Both Heegaard surfaces and incompressible surfaces play important roles in the study of 3-manifold topology, and there are some intriguing connections between them, for example, see [?, ?]. In [?], Hartshorn proved that if the distance $d(V, W)$ is large, then $M$ contains no small-genus incompressible surface. A natural question is whether there is a 3-manifold $M$ that has large Heegaard distance but contains no incompressible surface at all, i.e., $M$ is non-Haken (or small). A positive answer to this question is expected, but it is surprisingly hard to construct a concrete example.

There have been many constructions of non-Haken 3-manifolds, for example [?, ?, ?]. However, in some sense, the 3-manifolds in these constructions are relatively simple, but large Heegaard distance usually means that the 3-manifold is complicated. On the other hand, many complicated 3-manifolds do contain incompressible surfaces, e.g. [?, ?].

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In this paper, we construct the first examples of closed non-Haken hyperbolic 3-manifolds with a Heegaard splitting of arbitrarily large distance. Note that by a theorem of Scharlemann and Tomova (also see [9, 10]) if the distance of a Heegaard splitting $M = V \cup_F W$ is sufficiently large, then any minimal-genus Heegaard surface is isotopic to $F$.

**Theorem 1.1.** For any $g$, there are closed orientable non-Haken 3-manifolds with a genus-$g$ Heegaard splitting of arbitrarily large distance.

The motivation for this research is to explore some deeper relation between Heegaard surfaces, incompressible surfaces and the geometry of 3-manifolds. Many interesting questions in 3-manifold topology concerning non-Haken 3-manifolds are still open and it is possible that one can use Heegaard splittings to study these questions.

Our main construction is a combination of the constructions in [9] and [10]. In [9], Agol constructed a small link which is a pure braid in $S^2 \times S^1$. We basically take Agol’s construction in $S^2 \times I$ and consider its double branched cover. The double branched cover is homeomorphic to $F \times I$ and we use it as a neighborhood of our Heegaard surface $F$. In our construction, we start with a standard Heegaard splitting of $S^3$. Then we perform Dehn surgery on a link in $F \times I$, where $F$ is the Heegaard surface and the link is obtained by a Dehn twist on curves in a double branched cover of Agol’s construction in [9]. To guarantee the Heegaard distance is large, we use the construction and criterion given by Lustig and Moriah [11] on Heegaard distance.

We organize the paper as follows. In section ??, we briefly explain Agol’s construction and show that this construction also works for an immersed surface which lifts to an incompressible surface in a double branched cover. In section ??, we review some results in [9]. We finish the proof of Theorem ?? in section ??.

## 2. Agol’s construction and its double branched cover

We first briefly review a construction of Agol in [9]. The operation in [9] is on a pure braid in $S^2 \times S^1$, or equivalently, on $\Sigma \times S^1$, where $\Sigma$ is the $n$-punctured sphere. In this paper, we only consider $\Sigma \times I$. We use the same notations as those in [9] and refer the reader to [9] for more detailed discussions and proofs.

As in [9], we start with a pants decomposition of $\Sigma$ and use a special path in the pants decomposition graph $\mathcal{P}(\Sigma)^{(1)}$ (see [9, Definition 2.1]).

Let $D_n$ be a regular $n$-sided polygon and $\gamma$ its boundary. As in [9], we view $\Sigma$ as a 2-sphere obtained by gluing two copies of $D_n$ along $\gamma$, with punctures at the $n$ vertices. We cyclically label the $n$ edges of $D_n$ by $1, \ldots, n$. For each pair of edges $\{i, j\}$, there is a loop $\beta_{i,j}$ in $\Sigma$ which meets $\gamma$ exactly twice at the edges $i$ and $j$.

We start with an initial pants decomposition of $\Sigma$ given by a set of loops $P_0 = \{\beta_1, \beta_3, \beta_4, \ldots, \beta_{n-1}\}$. Let $C_0$ be a path of pants decompositions in $\mathcal{P}(\Sigma)^{(1)}$ denoted by $C_0 = P_0 \rightarrow P_1 \rightarrow \cdots \rightarrow P_{n-3}$, where