THE OPTION PRICING MODEL AND THE RISK FACTOR OF STOCK*

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In this paper a combined capital asset pricing model and option pricing model is considered and then applied to the derivation of equity's value and its systematic risk. In the first section we develop the two models and present some newly found properties of the option pricing model. The second section is concerned with the effects of these properties on the securityholders of firms with less than perfect 'me first' rules. We show how unanticipated changes in firm capital and asset structures can differentially affect the firm's debt and equity. In the final section of the paper we consider a number of theoretical and empirical implications of the joint model. These include investment policy as well as the causes and effects of non-stationarity in the systematic risk of levered equity and risky debt.

1. Introduction

The basic premise of this paper is that combining the option pricing model (OPM) with the capital asset pricing model (CAPM) yields a theoretically more complete model of corporate security pricing. From this vantage point we focus upon the issue of risk in corporate stock. We show that this synthesis of models leads to a number of insights regarding stock risk and changes in corporate asset structure and capital structure. In the process, we consider some important issues in corporate finance, illustrating the analytical advantages of this combined pricing model. Among these advantages is the ability to treat many of the issues in the corporate finance literature in a consistent and unified manner that can be easily quantified. Essentially, this paper is an attempt to gain a clearer focus, both theoretically and empirically, on the question of corporate stock risk and how the OPM adds to its understanding.

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Many of the theoretical results concerning the OPM used in our paper have their origin in Black-Scholes (1973) and Merton (1973a and 1974).
To simplify the analysis, we will consider a firm with one pure-discount bond issue and one common stock issue. The bond with face value C will mature at T (i.e., T periods from the current period which is denoted by 0) and at that time, the firm will liquidate itself. Up to T, the firm does not experience any net cash flows and pays no dividends to its shareholders. Under this set of simplifying assumptions, Black-Scholes (1973) observed that common stock can be regarded as a European call option.

After listing the assumptions needed, the capital asset pricing model and the option pricing model are presented in their continuous time framework. We explain how the firm's equity can be viewed as a call option when the underlying asset is the firm. From this perspective, the CAPM and the OPM yield the equilibrium value and expected rate of return of the firm and its equity (and debt) simultaneously. The implications of this interpretation of equity will be illustrated by a number of case studies, considering differences in firm asset and capital structures. In addition, we consider some portfolio rebalancing rules for firms, which protect the interests of all its securityholders. The final part will be devoted to implications - theoretical and empirical - of pricing corporate stock.

2. The assumptions

Under the following set of assumptions both the capital asset pricing model and the option pricing model can be derived.

(a) All individuals have a strictly concave von Neuman-Morgenstern utility function and are expected utility maximizers.
(b) There are homogeneous expectations about the dynamics of firm asset values and of security prices.
(c) The capital markets are perfect: there are no transaction costs or taxes and all traders have free and costless access to all available information. Traders are price takers in the capital markets, i.e., they are atomistic competitors.
(d) There are no costs of voluntary liquidation or bankruptcy, e.g., court or reorganization costs, where bankruptcy is defined as the state when the value of the firm's assets is less than the face value of the maturing debt.

3See Masulis (1975) for an application of the OPM to firms with more complex capital structures.
4Those familiar with the CAPM and the OPM may prefer to go directly to section 5 entitled 'Risk of equity', ignoring the description of the models in the preceding two sections.
4Note that this set of assumptions is a sufficient set; specifically, assumptions (a) and (b) are not required for the derivation of the OPM. Some assumptions are stronger than needed. Merton, for example, proves that the CAPM can be derived for a more general case where the dynamics of the price change can be described by the instantaneous expected rate of return \( \bar{r} \) and the instantaneous standard deviation of return \( \sigma \), and a simple Gauss-Wiener process (Gaussian 'white noise'). The parameters \( \bar{r} \) and \( \sigma \) are not necessarily constant over time; if they are, then we have a log-normal distribution as assumed above [i.e., assumption (g)]. For further details, see Merton (1973b).
(e) There is a known instantaneously riskless interest rate which is constant through time and is equal for borrowers and lenders.
(f) Borrowing and short-selling by all investors and free use of all proceeds are allowed.
(g) The distribution of firm asset value at the end of any finite time interval is log normal. The variance of the rate of return on the firm is constant.
(h) Trading takes place continuously, price changes are continuous and assets are infinitely divisible.

3. The capital asset pricing model and the valuation of the firm

It is implicit in the CAPM that investors differentiate assets only according to the assets' expected rates of return and their contribution to the variance of investors' efficient portfolios. According to the continuous time CAPM, the capital market will be in equilibrium only if at each instant of time assets are priced so that

\[ \tilde{r}_i = r_F + \beta_i(r_M - r_F). \]  

(1)

The instantaneous expected rate of return of asset \( i \), \( \tilde{r}_i \), is a linear function of its instantaneous systematic risk \( \beta_i \). The slope is determined by the instantaneous market risk premium \( (\tilde{r}_M - r_F) \) and the intercept is the instantaneous riskless interest rate \( r_F \), where \( \tilde{r}_M \) is the instantaneous expected rate of return on the market. The instantaneous systematic risk is defined as

\[ \beta_i \equiv \text{cov} (\tilde{r}_i, \tilde{r}_M)/\sigma^2(\tilde{r}_M), \]  

(2)

the instantaneous covariability of asset \( i \)'s percentage return with the percentage return on the market, standardized by the instantaneous variance of the market's percentage return. It should be noted that the instantaneous expected rate of return \( \tilde{r}_i \) is not a direct function of the instantaneous variance of the asset's rate of return. This variance includes non-systematic risk which can be costlessly diversified away; therefore, the market price for bearing this risk is zero.

We have assumed that our firm \( J \) expects to realize all its cash flows at the end

5 We do not interpret perfect capital markets as implying the existence of side payments between classes of securityholders or of perfect 'me first' rules. For our purposes, we define perfect 'me first' rules as rules restricting the firm's management from changing its asset or capital structure in any way that improves the value of one class of securities at the expense of another class. It should be obvious that perfect 'me first' rules would, in general, severely restrict the actions of a firm. For a further discussion of perfect competition in the capital market see Merton-Subrahmanyan (1974).

6 The derivation of the CAPM in a discrete time framework can be found in Sharpe (1963 and 1964), Lintner (1965a and 1965b), Mossin (1966), and Fama-Miller (1972, chs. 6 and 7); and in a continuous time framework in Merton (1970 and 1973b). Jensen (1972) summarizes the different approaches and provides a survey of empirical tests of the model.
of a discrete period of length $T$. Given the finite life of the firm, the equilibrium present value of the firm can be written as follows:

$$V^I_0 = \left[ V_T^I - \frac{\lambda \, \text{cov}(V_T^I, P_T^M)}{\sigma(P_T^M)} \right] / (1 + R_F).$$

(3)

The present value of firm $J$, $V^I_0$, is equal to the expected terminal value of the firm $V_T^I$ minus a premium for bearing non-diversifiable risk, all discounted at the discrete time riskless rate of return $R_F$, where $P_T^M$ is the market value at $T$ of the aggregate value of all risky firms (assuming all of them liquidate their assets at $T$), and $\text{cov}(V_T^I, P_T^M)$ is the covariance of firm asset value with total market value over $T$. A unit of risk is measured by $\text{cov}(P_T^I, P_T^M)/\sigma(P_T^M)$; and $\lambda$, which is the market price per unit of risk, is defined by $(R_M - R_F)/\sigma(R_M)$, where $R_M$ is defined as the discrete time expected market rate of return. Note that while we assume the firm's cash flow is discrete, trading in the firm's securities is continuous throughout the period.

4. The option pricing model and the valuation of equity

The option pricing model as derived by Black-Scholes (1973) applies to European-type options. They create a perfect hedge, at each instant of time, composed of one unit long (short) of the underlying security and a short (long) position on a number of options. The return on a completely hedged position should be equal to the riskless return on the investment in order to eliminate arbitrage opportunities. The resulting value for a European call option is

$$S = VN(d_1) - C e^{-r_T} N(d_2),$$

(4)

where $V$ is the current value of the underlying asset, $\sigma^2$ is the instantaneous variance of percentage returns on $V$, $C$ is the exercise price of the option, $T$ is the time to maturity, $r_F$ is the riskless interest rate, $N(\cdot)$ is the standardized normal cumulative probability density function, and

$$d_1 = \frac{\ln(V/C) + (r_F + \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}}.$$

$R_F$ is the instantaneous riskless rate of return $r_F$, continuously compounded over period $T$. $R_M$ is the instantaneous expected market rate of return $r_M$, continuously compounded over period $T$. European-type options are options that cannot be exercised before the expiration date. For a basic description of options see Kruizenga (1967) and Galai (1974a and 1974b). Assuming the firm's asset value is unaffected by its capital structure it can be shown that the debt of our firm has the value

$$D = V - S = VN(-d_1) + C e^{-r_T} N(d_2).$$
and
\[ d_2 = d_1 - \sigma \sqrt{T}. \]

For the kind of firm which we have hypothesized above, it was shown by Black–Scholes (1973) that the firm's equity can be regarded as a European call option. To see this note that the owner of a call option has claim to the slice of a stock's price distribution to the right of the exercise price at maturity date \( T \). Similarly, a firm’s stockholders have claim to the slice of the firm’s price distribution to the right of the face value of the firm’s debt at its maturity date. To complete the analogy we view the stockholders as having an option to buy back the firm (whose current market value is \( V \)) from the bondholders for an exercise price equal to the face value of the firm’s debt \( C \) at time \( T \). If the value of the firm at maturity \( V_T \) is above \( C \), the equity will have a positive value; if it is below, the stock is valueless. In other words, the stockholders have protection against depreciation of the firm’s value below \( C \) (this is the 'limited liability' nature of equity) and have a right to the appreciation in the firm’s value above \( C \).

We can apply the comparative static results derived for call options by Black–Scholes (1973) and Merton (1973a) for our model to show the effect of the parameters in eq. (4) on the value of the stock. It can be shown that

\[ 1 \geq \frac{\partial S}{\partial V} \geq 0, \quad \frac{\partial S}{\partial C} < 0, \quad \frac{\partial S}{\partial r_f} > 0, \quad \frac{\partial S}{\partial \sigma^2} > 0, \quad \frac{\partial S}{\partial T} > 0. \]

In words, the value of the stock is an increasing function of the value of the firm, the riskless interest rate, the variance of the percentage return of the firm and the time to liquidation; and it is a decreasing function of the face value of the debt.\(^1\)

The above formulation requires only partial equilibrium. Given the current market value of the firm \( V \), eq. (4) tells us the equilibrium value of the equity; however, this does not require that \( V \) be the equilibrium value of the firm. Nevertheless, as was indicated above, we can find the equilibrium value of the firm using the CAPM. It should be noted again that while \( S \) is a function of the firm’s current market value and the variance of the firm’s rate of return, it is not directly a function of the firm’s expected rate of return or systematic risk, as will be explained in the next section.

\(^1\)See appendix I for actual values of the partial derivatives. From the equation in footnote 10 it was shown by Merton (1974) that

\[ \frac{\partial D}{\partial V} = 1 - \frac{\partial S}{\partial V}, \quad \frac{\partial D}{\partial C} = -\frac{\partial S}{\partial C}, \quad \frac{\partial D}{\partial r_f} = -\frac{\partial S}{\partial r_f}, \quad \frac{\partial D}{\partial \sigma^2} = -\frac{\partial S}{\partial \sigma^2}, \quad \frac{\partial D}{\partial T} = -\frac{\partial S}{\partial T}. \]

\(^2\)For comparative static purposes these changes of variables can be either anticipated or unanticipated, but for dynamic analysis they must be unanticipated by the market.
5. The risk of equity

Now we will show that if the systematic risk of the firm $\beta_V$ is constant over time, the instantaneous risk of the equity $\beta_S$ will not necessarily be stable or known with certainty for the time period in question. Therefore, determining the current value of the equity from the CAPM, even when its expected value at the horizon point $T$ is known, is not a facile procedure.

From stochastic calculus and our assumptions, the dollar return on an option, and thus the dollar return on the equity, can be described as

$$\Delta S = S_v \Delta V + \frac{1}{2} S_{vV} \sigma^2 V^2 \Delta t + S_t \Delta t,$$

where $S_v \equiv \partial S/\partial V$, $S_{vV} \equiv \partial^2 S/\partial V^2$ and $S_t \equiv \partial S/\partial t$. Dividing $\Delta S$ by $S$ and substituting for $\bar{r}_S$ we obtain in the limit (as $\Delta t \to 0$)

$$\frac{\Delta S}{S} = S_v \frac{\Delta V}{V} \quad \text{or} \quad \bar{r}_S = S_v \frac{\Delta V}{V} \bar{r}_V. \quad (6)$$

Defining $\beta_S$ according to (2) and substituting into the instantaneous covariance term of definition (6) yields

$$\beta_S = \frac{\text{cov}(\bar{r}_S, \bar{r}_M)}{\sigma^2(\bar{r}_M)} = \frac{S_v}{S} \frac{\text{cov}(\bar{r}_V, \bar{r}_M)}{\sigma^2(\bar{r}_M)} \equiv \frac{S_v}{S} V \beta_V. \quad (7)$$

In words, the systematic risk of equity is the product of the firm's systematic risk and the elasticity of equity value with respect to firm value.

Taking the derivative of stock value with respect to firm value in the OPM equation (4), Black-Scholes (1973) found that $S_v = N(d_1)$. So combining the CAPM with the OPM yields

$$\beta_S = N(d_1) \frac{V}{S} \beta_V \equiv \eta_S \beta_V. \quad (8)$$

13 Parts of the analysis here are based on Black-Scholes (1973).
14 See Black-Scholes (1973, eq. 15).
15 The partial derivative of the debt value with respect to the firm value is shown in the equation in footnote 11 to be

$$D_V = 1 - S_V = N(-d_1).$$

Repeating the arguments used to prove eq. (7) we obtain the relationship between the systematic risk of the bond and of the firm,

$$\beta_\sigma = D_v \frac{V}{D} \beta_V = N(-d_1) \frac{V}{D} \beta_V \equiv \eta_\sigma \beta_V. \quad (9)$$

16 Given $\sigma_\delta^2 = \text{cov}(\bar{r}_S, \bar{r}_S)$, we can also show that $\sigma_\delta = S_v(V/S) \sigma_V \equiv \eta_\delta \sigma_V.$
Substituting eq. (4) into the definition of the elasticity term \( \eta_S \) in eq. (8), we obtain

\[
\eta_S = \frac{VN(d_1)}{S} = \frac{VN(d_1)}{VN(d_1) - C e^{-rT}N(d_2)} = \frac{1}{1 - (C/V) e^{-rT}N(d_2)/N(d_1)}.
\]

The limited liability characteristic of options \( 0 \leq S = VN(d_1) - C e^{-rT}N(d_2) \) implies

\[
\frac{C e^{-rT}N(d_2)}{VN(d_1)} \leq 1,
\]

so the denominator of the right-hand term of the definition of \( \eta_S \) is less than one. Since \( \eta_S \geq 1 \), the systematic risk of equity is greater than or equal to the systematic risk of the firm (for \( \beta_v > 0 \)).

In the case where the firm's systematic risk is stationary, the implication of eq. (8) is that its equity's systematic risk will generally be non-stationary. That is, for the vector of parameters \( V, C, r_F, \sigma^2 \) and \( T \) denoted as \( K \),

\[
\frac{\partial \beta_S}{\partial K} = \frac{\partial \eta_S}{\partial K} \beta_v + \frac{\partial \beta_v}{\partial K} \eta_S.
\]

But by the assumption of stationarity for \( \beta_v \), we then obtain

\[
\frac{\partial \beta_S}{\partial K} = \frac{\partial \eta_S}{\partial K} \beta_v.
\]

We have proved that (when \( \beta_v > 0 \))

\[
\frac{\partial \beta_S}{\partial V} < 0, \quad \frac{\partial \beta_S}{\partial C} > 0, \quad \frac{\partial \beta_S}{\partial r_F} < 0, \quad \frac{\partial \beta_S}{\partial \sigma^2} < 0, \quad \frac{\partial \beta_S}{\partial T} < 0.
\]

The analysis indicates that the relationship between the systematic risk of the firm \( \beta_v \) and of its equity \( \beta_S \) is not only a positive function of the firm's leverage
V/S as shown by Hamada (1972), but that it is a positive function of the face value of debt C and a negative function of the value of the firm V, the riskless interest rate \( r_f \), the variance of the firm's percentage returns \( \sigma^2 \) and the time to maturity of the firm's debt \( T \). Since \( \beta_S \) is a function of the time to maturity of the debt and the realizations of \( V \) at each instant, it will usually change from instant to instant.

The factors determining the expected rate of return on equity have been extensively analyzed in the existing literature. A few of the more important results will be reinterpreted and extended below, utilizing the option characteristics of equity.

The instantaneous expected rate of return of a firm is equal to the instantaneous expected rates of return of its securities (debt and equity) weighted by the relative value of their claims on the firm,

\[
\bar{r}_V = \frac{S}{V} \bar{r}_S + \frac{D}{V} \bar{r}_D, \tag{11}
\]

so that

\[
\bar{r}_S = \bar{r}_V + \left[ \bar{r}_V - \bar{r}_D \right] \frac{D}{S}. \tag{12}
\]

Eq. (12) is proposition II of Modigliani–Miller generalized to risky debt where the CAPM has replaced the risk class assumption. Defining \( \bar{r}_S \) from eq. (1) and substituting for \( \beta_S \) from eq. (8) and for \( \beta_V = (\bar{r}_V - r_f)/(\bar{r}_M - r_f) \) from eq. (1), we obtain an alternative expression of the instantaneous expected rate of return on equity,

\[
\bar{r}_S = r_f + N(d_1)\left[ \bar{r}_V - r_f \right] \frac{V}{S}. \tag{13}
\]

19 \( \beta_S = (V/S)\beta_V \) (Hamada's equation 4a) which assumed that the debt was riskless. Hence, eq. (8) is a generalization of the Hamada result.

20 It is also shown in appendix I that for \( \partial \beta_V / \partial K = 0 \) and \( \beta_V > 0 \),

\[
\frac{\partial \beta_D}{\partial V} < 0, \quad \frac{\partial \beta_D}{\partial C} > 0, \quad \frac{\partial \beta_D}{\partial r_f} < 0, \quad \frac{\partial \beta_D}{\partial \sigma^2} > 0, \quad \frac{\partial \beta_D}{\partial T} < 0, \quad \frac{\partial \beta_D}{\partial g} < 0.
\]

These results are consistent with Merton's (1974) results,

\[
\frac{\partial \beta_D}{\partial \sigma^2} T \leq 0 \quad \text{and} \quad \frac{\partial \beta_D}{\partial g} > 0, \quad \text{where} \quad g = \frac{C e^{-r_f T}}{V}.
\]

21 The instantaneous expected rate of return defined in eq. (1) holds for any asset including options.

22 See Modigliani–Miller (1958) and Fama–Miller (1972, ch. 4).

23 For riskless debt \( \bar{r}_D = r_f \), which substituted into eq. (12) yields Hamada's (1969) equation 13. Merton (1974) shows eq. (12) to be a concave function of \( D/S \).
Rubinstein (1973) interprets the first term on the right-hand side of eq. (12) to represent the expected rate of return for the operating risk and the second term to represent the financial risk of a levered firm borne by its stockholders. Eq. (13) can also be written as

$$\tilde{r}_S = \tilde{r}_V + (\tilde{r}_V - r_F)(\eta_S - 1),$$

and the second term on the right-hand side of the equation stands for that part of the shareholder's return due to financial risk. Hence the expected rate of return due to financial risk can be written as follows:

$$(\tilde{r}_V - r_F) \frac{D}{S} = (\tilde{r}_V - r_F)(\eta_S - 1) = (\tilde{r}_V - r_F) \frac{C e^{-\tau T} N(d_2)}{S}.$$

From this expression we can see explicitly the terms that contribute to the higher required expected rate of return by stockholders due to leverage.

By using eq. (1) with our previous results for $\beta_S$ in (10) we can show that

$$\frac{\partial \tilde{r}_S}{\partial V} < 0, \quad \frac{\partial \tilde{r}_S}{\partial C} > 0, \quad \frac{\partial \tilde{r}_S}{\partial \sigma^2} \approx 0, \quad \frac{\partial \tilde{r}_S}{\partial \tau} < 0, \quad \frac{\partial \tilde{r}_S}{\partial \tau} < 0,$$

for the instantaneous expected rate of return on equity.$^{24}$

In the next section we will utilize the above analysis to explore a number of important questions in corporate finance. The implications of this analysis for empirical investigations of the CAPM will be emphasized in the last section.

6. Case studies

Throughout these case studies we will take a comparative-static approach. To do this we will first compare two, initially identical, firms ($A$ and $B$) after changing one or more of firm $B$'s relevant characteristics. The comparative firm analysis will be accompanied by numerical examples which can help to explain some observed differences in equity across firms. This will be followed by a comparative-static analysis of a single firm where firm $B$ will now represent firm $A$ at a second point in time.

We will consider, in the course of each case study, the effects of unanticipated changes in specific variables upon the systematic risk, the expected rate of return and the market value of a single firm's debt and equity. The analysis highlights the potential for a redistribution of wealth from one security class to another.

$^{24}$For $\beta_v < 0$ the signs of the partial derivative are reversed, with the exception that $\frac{\partial \tilde{r}_S}{\partial \tau} > 0$. 
when perfect 'me first' rules\(^5\) or side payments between security classes are non-existent or prohibitively expensive. From this analysis, we can better understand the motivations for indenture restrictions\(^6\) and a number of firm asset and capital structure changes.

Strictly speaking, while these redistribution effects exist in the Sharpe–Lintner CAPM, they will be irrelevant. This follows from the property of the CAPM that all investors hold the market portfolio; therefore, all investors hold equal proportions of each firm's debt and equity. Consequently, shifts of wealth from one class of securities to another leave all investors indifferent.\(^7\) Thus, protection or 'me first' rules are not needed and serve no economic purpose under these conditions. Such indifference to redistributions will not exist if investors do not all hold the market portfolio.\(^8\) We believe that our comparative-static analysis can, despite its limitations, serve a useful purpose in highlighting some of the potential effects of alternative corporate policies.

We will begin by comparing two levered firms (\(A\) and \(B\)) which in each case study differ in one or more relevant variables. The two firms have the same liquidation date, which is \(T\) periods from now, and at that time the pure discount bonds of both will mature. The parameters of the firms \(A\) and \(B\) are given in table I. Throughout the discussion tildes will denote stochastic variables and bars will denote expected values of variables.

<table>
<thead>
<tr>
<th>Variables of the firm</th>
<th>Firm A</th>
<th>Firm B</th>
<th>General</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current market value of firm</td>
<td>(V_0^A)</td>
<td>(V_0^B)</td>
<td>(V_0)</td>
</tr>
<tr>
<td>Terminal market value of firm</td>
<td>(V_T^A)</td>
<td>(V_T^B)</td>
<td>(V_T)</td>
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<tr>
<td>Current market value of shares</td>
<td>(S_0^A)</td>
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<tr>
<td>Current market value of debt</td>
<td>(D_0^A)</td>
<td>(D_0^B)</td>
<td>(D_0)</td>
</tr>
<tr>
<td>Systematic risk of firm</td>
<td>(\beta_A^S)</td>
<td>(\beta_B^S)</td>
<td>(\beta^S)</td>
</tr>
<tr>
<td>Systematic risk of shares</td>
<td>(\beta_A^S)</td>
<td>(\beta_B^S)</td>
<td>(\beta^S)</td>
</tr>
<tr>
<td>Variance of rate of return of the firm</td>
<td>(\sigma_A^2)</td>
<td>(\sigma_B^2)</td>
<td>(\sigma^2)</td>
</tr>
<tr>
<td>Rate of return of the firm</td>
<td>(r_A^R)</td>
<td>(r_B^R)</td>
<td>(r^R)</td>
</tr>
<tr>
<td>Rate of return of the shares</td>
<td>(r_A^S)</td>
<td>(r_B^S)</td>
<td>(r^S)</td>
</tr>
<tr>
<td>Face value of debt maturing at (T)</td>
<td>(C_A)</td>
<td>(C_B)</td>
<td>(C)</td>
</tr>
</tbody>
</table>

Case I. Rate of return variability and changes due to acquisitions and divestitures

Assume that

\[
C_A = C_B,
\]

\(^5\) See footnote 5 for a clarification of this assumption and the definition of 'me first' rules.

\(^6\) For an earlier qualitative analysis of this conflict of interest among securityholders, see Fama–Miller (1972, pp. 150–156, 178–180).

\(^7\) For that matter, shifts of wealth between firms also leave investors indifferent.

\(^8\) This is true of the Mayers (1973) CAPM with non-marketable human capital, which otherwise exhibits the major properties of the Sharpe–Lintner CAPM.
(b) \( V^A_t = V^B_t \),

(c) \( \text{cov} (\hat{V}^A_t, \tilde{V}^M_t) = \text{cov} (\hat{V}^B_t, \tilde{V}^M_t), \quad 0 \leq t \leq T. \)

\( \tilde{V}^M_t \) is defined at the end of section 3, and

(d) \( \sigma^2_A > \sigma^2_B. \)

From assumptions (b) and (c), and using eq. (3), we find that the market value of the two firms is identical, \( V^A_0 = V^B_0. \) The two firms are in the same risk class with the same systematic risk. They differ in their total variability of returns. How will this affect the market value of the shares of firms A and B?

If \( \sigma^2_A > \sigma^2_B \), then we can prove, using the OPM, that \( D^A_0 < D^B_0 \) and \( S^A_0 > S^B_0. \) The proof is based on the fact that options' values will be an increasing function of the variance of the underlying security, ceteris paribus. With the exception of rate of return variability, the parameters that determine the equity value of firms A and B are identical in terms of eq. (4) [note, however, that the equities' co-variability with the market is not assumed to be the same]. We showed before that equity can be regarded as a call option, and thus we can apply the result \( \partial S_0 / \partial \sigma^2 > 0 \) directly.\(^{30}\) We conclude that firms with apparently similar characteristics with regard to face value of debt, total market value and profitability, but with a different variance of rate of return, will have a different capital structure in market value terms. The market value of the debt-equity ratio \( (D/S) \) will be greater for the firm with lower variance. In our example, it will be true that \( D^A_0/S^A_0 < D^B_0/S^B_0. \)

To see the effect more clearly, we will assume that

\[
V^A_0 = V^B_0 = \$1,000,
C_A = C_B = \$500,
\sigma^2_A = 0.10 (10\%), \quad \sigma^2_B = 0.05 (5\%),
R_F = 0.08 (8\%).
\]

Then, for \( T = 5 \) (e.g., five years) we find, using eq. (4), that

\[
S^A_0 = \$675, \quad S^B_0 = \$666,
\]

\(^{4}\)See the derivatives in eq. (5) and footnote 11.
\(^{30}\)For the derivation see appendix 1.
and therefore,

\[ D_0^A = \$325, \quad D_0^B = \$334. \]

If we raise the variance of the rate of return of firm \( A \) to \( \sigma_A^2 = 0.15 \), then

\[ S_0^A = \$688, \quad D_0^A = \$312. \]

Alternatively, if we lowered the face value of the debt of both firms to \( \$400 \), we obtain (for \( \sigma_A^2 = 0.10 \) and \( \sigma_B^2 = 0.05 \))

\[ S_0^A = \$736, \quad S_0^B = \$732, \]

\[ \text{and} \quad D_0^A = \$264, \quad D_0^B = \$268. \]

Differences in the variance of the rate of return of firms cause differences in the market value of the firm's equity and debt, and such differences can be quantified (at least for our simplified world). We see from the last example that the effect of such a difference in the variance on the value of the debt and equity is diminished by a decline in the debt–equity ratio.

Given the assumptions of this case study another prediction can be made with respect to the differences in the expected rates of return on the shares of the two firms. We previously showed that \( \partial \beta / \partial \sigma^2 < 0 \), so we can expect to obtain lower rates of return on the equity of firms with larger rate of return variances \( \sigma^2 \) (i.e., firm \( A \) in our example). So the value of a firm's equity will be higher while its expected rate of return will be lower as a function of the firm's rate of return variance \( \sigma^2 \), ceteris paribus.

In the context of a single firm, consider management making an unanticipated acquisition, divestiture or other investment decision which changes the variance of the firm's rate of return. To keep the presentation simple, view an acquisition as an exchange of riskless assets (riskless government securities) for risky physical assets, and a divestiture as just the reverse. Then it should be obvious that in a world of imperfect 'me first' rules, such an unanticipated investment decision will indeed change the market values of the firm's debt and equity.

**Case II. Changes in the scale of a firm and the problem of dilution**

Assume that

\[ P_t^A = \alpha P_t^B, \quad 0 \leq t \leq T. \]

This implies

\[ P_T^A = \alpha P_T^B. \]

\(^{31}\text{For firms with positive systematic risk, see appendix I.}\)

\(^{32}\text{Assume for simplicity that the assets acquired or divested are economically independent of the other assets of the firm.}\)
and
\[ \text{cov}(\bar{P}^A_t, \bar{P}^M_t) = \alpha \text{cov}(\bar{P}^B_t, \bar{P}^M_t), \quad 0 \leq t \leq T, \]

which together with the valuation eq. (3) yields

\[ V^A_0 = \alpha V^B_0. \]

Assumption (a) also implies that the two firms' rates of return have perfect positive correlation and therefore

\[ \sigma^2_A = \sigma^2_B, \]

and

\[ \beta^A_V = \beta^B_V. \]

If we further assume that

\[ C_A = \alpha C_B, \]

then from eq. (4) we see that \( d_1^A = d_1^B \) and \( d_2^A = d_2^B \), so

\[ S^A_0 = \left( \alpha V^B_0 \right) N(d_1^B) - \left( \alpha C_B \right) e^{-rT} N(d_2^B) = \alpha S^B_0. \]

Hence, if two firms are identical except that they differ by the same proportion in terms of firm asset value and face value of debt, then their equities' (debts') value will also differ by this proportion. Using eq. (8) and then substituting in the above relationship yields

\[ \beta^A_S = \frac{\alpha V^B_0}{\alpha S^B_0} N(d_1^B) \beta^B_V = \beta^B_S. \]

The systematic risk of the two firms' debt and equity are identical. They are unaffected by the proportional differences in the two firms.

We can reach the same conclusion for an unanticipated change in the scale of an individual firm, externally financed.\footnote{For simplicity we assume stochastic constant returns to scale, which is consistent with a perfectly competitive capital market as shown by Merton-Subrahmanyam (1974).} From the option pricing model alone, it can easily be shown that the value of equity and debt can be written as\footnote{Note that} \( S = S_V V + S_C C \),

\[ (17) \]

\[ S = S_V V + S_C C, \]
and
\[ D = D_V V + D_C C, \]  
(18)
both of which are first-degree homogeneous functions of \( V \) and \( C \). It can also be shown from the OPM that the systematic risk of the equity and debt are zero-degree homogeneous functions of \( V \) and \( C \), hence
\[ \frac{\partial^2 \beta_S}{\partial V} V + \frac{\partial^2 \beta_S}{\partial C} C = 0, \]  
(19)
and
\[ \frac{\partial^2 \beta_D}{\partial V} V + \frac{\partial^2 \beta_D}{\partial C} C = 0. \]  
(20)
The content of the above results is that there is a financing policy devoid of redistribution effects. For a proportional rise in the firm’s scale of operations of \( \Delta V/V \), the firm can issue debt until \( \Delta C/C = \Delta V/V \) and then raise the remaining capital with new equity. This is equivalent to increasing the firm’s debt and equity proportionately with the rise in the firm’s scale. If the firm’s unanticipated expansion is financed in any other combination of debt and equity, there will be a ‘watering down’ or dilution effect on one or the other class of securities.  

Case III. Conglomerate mergers

In this case, we want to investigate the effects of a pure conglomerate merger, in a perfect capital market, on the values of the equity and debt of the two firms that are involved. Because the merger is defined as a conglomerate type, we are assuming that there is no economic ‘synergy’ effect. Merger of two firms with less than a perfect correlation of their returns will decrease the variance of the new firm (assuming initially, without loss of generality, that \( \sigma^2_A = \sigma^2_B \)), and thus reduce the value of the unprotected equity and increase the market value of debt.

We will assume that firm \( G \) owns exactly the same assets as held by firms \( A \) and \( B \) and that there is no economic dependence between the assets of the two firms. Specifically, we assume
\[ \tilde{V}_t^G = \tilde{V}_t^A + \tilde{V}_t^B, \quad 0 \leq t \leq T, \]
(33)See appendix I for the partial derivatives.

\[ \text{Since } \Delta D/D = \Delta C/C, \text{ then } \Delta D/D = \Delta V/V \text{ which implies } \Delta S/S = \Delta V/V. \]
\[ \text{If the expansion is financed with only equity } \Delta V = \Delta S, \text{ the old stock will be diluted. If the expansion is financed with debt (of the same seniority) the old debt's value is diluted. An equity-financed expansion decreases the systematic risk of the debt and equity since } \frac{\partial \beta_D}{\partial V} < 0 \text{ and } \frac{\partial \beta_D}{\partial C} < 0. \text{ A debt-financed expansion increases the systematic risk of the debt and equity since } \frac{\partial \beta_S}{\partial C} > 0 \text{ and } \frac{\partial \beta_D}{\partial C} > 0, \text{ and this dominates the effect of a rise in } V \text{ for } \Delta C/C > \Delta V/V, \text{ by the zero-degree homogeneity of eqs. (19) and (20) with respect to } V \text{ and } C. \]
\[ \text{See Levy–Sarnat (1970), Lewellen (1971) and Lintner (1971).} \]
which can be seen from the analysis of Case II to imply

\[(a') \quad V^G_0 = V^A_0 + V^B_0,\]

and

\[(a') \quad \beta^G_v = \gamma \beta^A_v + (1 - \gamma) \beta^B_v, \quad \text{where} \quad \gamma = \frac{V^A_0}{V^G_0},\]

\[(b) \quad C_G = C_A + C_B,\]

\[(c) \quad \rho(\bar{\alpha}^A_v, \bar{\alpha}^B_v) < 1,\]

where \(\rho\) is the correlation coefficient.

For expositional simplicity we will further assume

\[(d) \quad \sigma^2_A = \sigma^2_B,\]

\[(c) \quad V^A_0/C_A = V^B_0/C_B.\]

Assumptions (c) and (d) imply that

\[(f) \quad \sigma^2_G < \sigma^2_A = \sigma^2_B,\]

while assumptions (a'), (b) and (c) yield

\[(g) \quad V^G_0/C_G = V^A_0/C_A = V^B_0/C_B.\]

From the results (f) and (g) combined with the analysis of Case I, we see that eq. (4) implies

\[S^G_0 < S^A_0 + S^B_0 \quad \text{and} \quad D^G_0 > D^A_0 + D^B_0.\]

The risk of ruin of firm \(G\) is smaller than that facing \(A\) or \(B\) separately (\(\sigma^2_G < \sigma^2_A = \sigma^2_B\)). Therefore the market value of firm \(G\)'s bonds is greater than the sum of the market values of the bonds of firms \(A\) and \(B\). Their promised terminal values are the same as shown in (b) above. On the other hand, the market value of firm \(G\)'s stock is smaller than the sum of the values of firms \(A\) and \(B\)'s stock by an equal amount.

This analysis can be applied to the case of a conglomerate merger. If investors are unprotected against changes in the volatility of their holdings, the value of

\[3^* \text{Since } \sigma^2_G = \gamma^2 \sigma^2_A + (1 - \gamma)^2 \sigma^2_B + 2(1 - \gamma) \sigma_A \sigma_B \rho(\bar{\alpha}^A_v, \bar{\alpha}^B_v), \quad \text{where} \quad \gamma = \frac{V^A_0}{V^G_0}.\]

\[4^0 \text{From assumptions (a) and (b) we have } V^G_0/C_G = \delta (V^A_0/C_A) + (1 - \delta)(V^B_0/C_B), \quad \text{where} \quad \delta = \frac{C_A}{C_G}.\]
their holdings might be changed. It is assumed here that each bond of the two original firms is exchanged for a bond of identical face value, with the same seniority and maturity, and guaranteed by the new firm. This assumption, which will be discussed further at a later point, is made in order to emphasize the concept of 'debt capacity' in the firm. Stock in the new firm is distributed according to the relative equity value of the two firms before the merger is announced. Under the above assumptions the stockholder's position can be expected to deteriorate with the unanticipated announcement of a merger between $A$ and $B$, due to the lower variance of the new firm's (denoted hereafter by $G$) rate of return. The bondholders of the merged firm $G$ are better off since the risk of bankruptcy has decreased. What is taking place, as Rubinstein (1973) points out, is that the bondholders receive more protection since the stockholders of each firm have to back the claims of the bondholders of both companies. The stockholders are hurt since their limited liability is weakened. An alternative solution to this refinancing problem is to retire the existing debt of firms $A$ and $B$ at their market value (assuming the market anticipates no redistribution effects) and then to issue debt in firm $G$ with a market value equal to the preexisting debt of firms $A$ and $B$. Other solutions are also possible.

In our world of no transaction costs of bankruptcy [assumption (d) of the OPM] there is no financial synergy which increases the value of the merged firm $G$ as Lewellen (1971) and Lintner (1971, p. 107) assert, nor any economies of scale associated with the cost of capital as suggested by Levy-Sarnat (1970, p. 801). This can be seen once one recognizes that investors in the marketplace could have created an identical financial position by purchasing equal proportions of the debt and equity of the two firms. The value of the sum of all the merged firm's liabilities equals the sum of its assets, the latter being a function of the firm's production-investment policy. But firm $G$'s production-investment policy is only the sum of the policies of the original two firms which are unchanged since there is no economic synergy in a pure conglomerate merger. So the value of firm $G$'s liabilities is simply the sum of the asset values of firms $A$ and $B$, which in turn are equal to the sum of their liabilities. This result does not necessarily hold for the value of debt or equity alone because they are functions of the volatility of the firm's returns, and the volatility is not an additive function when assets are not perfectly positively correlated. Hence, changes in the values of specific liabilities can be expected to occur under mergers.

This case describes a situation where securityholders (i.e., stockholders, in our specific example) do not have adequate protection against financial policy that can change their wealth. A more interesting question is how securityholders will be compensated so that they will have no incentive to block a conglomerate

\footnote{A similar, but rather qualitative, claim can be found in Higgins' comments (1971) to Lewellen's paper.}

\footnote{This point is proved by Levy-Sarnat (1970).}

\footnote{See Stiglitz (1969 and 1972), and Fama-Miller (1972, ch. 4, pp. 150-156).}
type merger. In our example above, one way to do this is by issuing more debt with the same seniority and retiring a certain fraction of the merged firm's equity. By doing so, the value of the original bonds will decline. This process can be continued until the original bondholders' holdings have a market value identical to their combined market value before the merger took place. The result of this process is an increase in the debt/equity ratio of the merged firm. In other words, by increasing the debt-equity ratio of the merged firm, the market values of the original securityholders can be restored to their pre-merger levels. This result is consistent with the claim that mergers 'allow' the firms to increase their 'debt capacity'. For some numerical examples of this type, see appendix II. As was mentioned previously, this process of refinancing is not unique - other alternatives also exist. Under our assumptions, the 'debt capacity' of the firm has increased, while the wealth of individual securityholders remains unchanged. In a world with corporate taxes where interest payments on debt are tax deductible, increases in 'debt capacity' increase the after tax value of the firm. This may help explain the motivations behind the conglomerate merger movement of recent years.

**Case IV. Spin-offs**

The obverse of a merger is a spin-off: the division of a single firm into two separate corporate entities. The conventional procedure is to take a portion of a firm's assets, often a division relatively unrelated to the remaining operations of the firm, and create a legally independent firm with these assets. The crucial facet of the procedure hinges on distributing the shares of this new equity solely to the stockholders of the parent corporation. In effect, the stockholders have 'stolen away' a portion of the bondholders' collateral since they no longer have any claim on the assets of the new firm.

To illustrate this we will assume

(a) \( \bar{V}_i^G = \bar{V}_i^A + \bar{V}_i^B \), \( 0 \leq t \leq T \).

implying no economic dependence between \( A \) and \( B \). Assumption (a) implies

(b) \( V_G^0 = V_A^0 + V_B^0 \).

We can view firm \( G \) as being composed of two economically independent divisions \( A \) and \( B \). At time 0 firm \( G \) unexpectedly spins off division \( B \), so that firm \( G \) is now composed solely of division \( A \). Hence,

(c) \( C_G = C_A \).

---

44See Lewellen (1971).
45Dividends can be treated similarly (as shown by Black-Scholes (1973)), as can the firm's repurchase of its own stock in the capital market (treasury stock).
46Note that in this case study we have gone directly to a comparative-static framework, omitting initial development of the comparative firm analysis.
As a result of the spin-off the debtholders of \( A \) (debtholders of firm \( G \) after the spin-off) find that their position has deteriorated because less assets now serve as collateral for the debt. Furthermore, the leverage \( V/C \) of the firm has gone up due to the loss in assets, so \( \beta_A^G > \beta_S^G \) and \( \beta_B^A > \beta_B^G \). Moreover, the variance of the firm's rate of return will, in general, change (\( \sigma_S^2 \neq \sigma_G^2 \)) due to the spin-off.\(^4\)\(^8\) This would give the additional results illustrated in Case Study 1. For simplicity, we will assume that this variance remains constant.

Hence, we observe that

\[ D_0^A \leq D_0^G, \]

which combined with the assumption (b) yields

\[ S_0^A + S_0^R > S_0^G. \]

In words, the value of the holdings of the equityholders of firm \( G \), who are now the equityholders of firms \( A \) and \( B \), will increase at the expense of firm \( G \)'s debtholders who are now the debtholders of firm \( A \). This is just another case where the lack of protection against investment and financial decisions of the firm by classes of securityholders may result in deterioration of their positions. The qualitative analysis in all of the above cases can be quantified and thus illustrate more powerfully the extent of deterioration in the positions of specific classes of securityholders.

If, in any of the above cases, the firm's decision had been anticipated by the market, there would be no redistribution effects. However, if the market over-anticipates the magnitude of the firm's change in policy, the redistribution effects among the classes of securityholders would be reversed.

7. An application to corporate investment decisions

One question not considered in our case studies is that of corporate investment decision making. We continue to assume in this section that no side payments or perfect 'me first' rules are allowed or that the transactions costs of affecting such actions are prohibitively large.\(^4\)\(^9\) Jensen-Long (1972) and Merton-Sabrman-yam (1974) proved that an unlevered firm in a perfectly competitive environment under uncertainty acts to maximize its current value. They implicitly, if not

\(^4\) See the partial derivatives for \( V \) in eq. (10) and footnote 20.

\(^8\) By substituting \( \sigma_A^2 \) for \( \sigma_S^2 \) in the equation

\[ \sigma_A^2 = 2 \sigma_A^2 + 2(1 - z) \sigma_A^2 + 2(1 - z) \text{cov} (r, r^u), \]

where \( z \equiv V_A^u, V_B^u \), we see that \( \sigma_A^2 \geq \sigma_A^2 \) if \( \sigma_A^2 \leq \sigma_A^2 + 2z \text{cov} (r, r^u) \).

\(^9\) See footnote 5 for a further discussion of this assumption.
explicitly, assumed that this result would also hold for levered firms. But in a world of imperfect 'me first' rules where the stockholders control the investment decisions of the firm, this may not be the case. Consider a firm which unexpectedly finds a new investment opportunity. It has a choice between two mutually exclusive projects of equal profitability in terms of expected net cash flow (discounted for systematic risk), but one project has a higher variance of percentage returns. Then from our earlier analysis, it should be clear that the firm controlled by its stockholders will invest in the project of higher variance. Moreover, it is even possible that a more profitable investment project will be rejected in favor of a project with a higher variance of percentage returns. While a pure equity firm will accept a project if the market value of the firm is increased by the investment \( dV/dl \geq 0 \), a levered firm will accept the project only if \( dS/dl = (\partial S/\partial V)(dV/dl) + (\partial S/\partial \sigma^2)(d\sigma^2/dl) \geq 0 \). One interpretation of this is that the cost of capital used in making the firm's investment decisions is a negative function of the change in the firm's rate of return variance if the investment is accepted. The reason why the levered firm does not maximize the market value of the firm is due to an externality affecting the securityholders of the firm.

For an unanticipated rise in the firm's variance of percentage returns due to a new investment project, there will be a fall in the value of the bonds and a rise in the value of the stock. This will also cause a rise in the systematic risk borne by the bondholders and a fall in that borne by the stockholders.

8. Implications for empirical studies of debt and equity

A number of empirical implications can be derived from our model. Most of these implications are based on the result that the systematic risk and rate of return variance of levered equity and risky debt are in general non-stationary. Hence, the rate of return distributions of this debt and equity will generally also be non-stationary. This will present a number of statistical difficulties in measuring security risk and in testing the efficiency of the capital market or the validity of the CAPM, which we will now detail.

Both the above studies utilized the simple CAPM which implies that everyone holds the market portfolio and therefore everyone holds an equal proportion of each firm's debt and equity. Hence there is no motive for affecting a redistribution of wealth without the introduction of a more 'realistic' asset pricing model.

For an earlier treatment of this possibility using a state preference model, see Fama-Miller (1972, pp. 178-181).

This assumes no change in firm scale. It is, rather, a change in asset composition, e.g., a change in the holdings of riskless government debt. For external financing or a dividend reduction the decision rules are as follows: for unlevered firms \( dV/dl \geq 1 \), and for levered firms \( dS/dl = (\partial S/\partial V)(dV/dl) + (\partial S/\partial \sigma^2)(d\sigma^2/dl) + (\partial S/\partial C)(dC/dl) \geq 1 \), where the third term only appears if there is some debt financing. This assumes that there is no effect on the total supply or composition of the capital market's assets.

Actually, this result is an example of the moral hazard problem [see Arrow (1970)].
There is a great deal of empirical evidence by Blume (1968 and 1971), Gonedes (1973), Bachrach-Galai (1974) and others indicating that individual common stock β's are non-stationary. Surprisingly, little empirical work has been done, utilizing information concerning changes in the firm's asset and capital structure, to predict changes in its securities' risk. One notable exception is Hamada (1972). Utilizing a model which allows a firm only riskless debt, Hamada found that changing a firm's leverage may cause the systematic risk of the stock to be non-stationary. The total firm's systematic risk may be stable (as long as the firm stays in the same risk class), whereas the common stock's systematic risk may not be stable merely because of unanticipated capital structure changes. He then went on to test this empirically and found that taking account of leverage improves the estimation of β (the variance of the estimates was lowered). This would seem to be only one of many possible sets of information concerning firm asset and capital structure changes which could help predict changes in the risk of individual securities.

In an efficient capital market, any new information reaching the market concerning asset values is immediately impounded into security prices. From our previous analysis of the variables causing redistribution effects, we should expect to find an empirical relationship between changes in security prices or in their systematic risk and the appearance of new information in the market concerning the variance of the firm's rate of return, the riskless interest rate, the time to maturity of its debt and the face value of debt to asset value ratio. So new information concerning changes in a firm's asset structure or financial structure as they affect the above variables should be seen to simultaneously change the prices and systematic risk of the firm's securities.

One implication of this is that one can expect on average that the realized rate of return on securities will be affected not only by changes in the expected terminal value but also by changes in their systematic risk. Unfortunately, this compounds the problem of measuring and interpreting the excess realized rates of return due to information effects such as the study done by Fama-Fisher-Jensen-Roll (1969). A more complete way of measuring the effects of information would be to devise a joint test of changes in systematic risk and excess realized rates of return. Aside from this methodological criticism of Fama-Fisher-Jensen-Roll, we also would like to suggest an alternative interpretation of their statistical results. They studied the information effects of stock splits and found that stocks which split and later had increases in dividends also had positive excess realized rates of return. They concluded that this shows that dividends give positive information about the firm to the market. From our earlier analysis, we would conclude that this phenomena may be due, at least in part, to the positive

54 The option pricing model, in addition, implies that changes in the firm's variance of percentage returns, the remaining life of the debt, and riskless interest rate also affect the systematic risk of a firm's debt and equity.

55 Hamada (1972, p. 443).
redistribution effect of the unanticipated dividend rise. Moreover, we would predict an adverse effect upon the value of the firm's debt while the information hypothesis would predict the reverse.

Much of the empirical work testing the efficient capital market assumption has assumed that the distribution of common stock returns behaves as a random walk. A necessary ingredient for this to be true is stationarity of the returns distribution. This, however, is generally not possible since the rate of return of common stock $\tilde{r}_S$ (in a levered firm) is a non-stationary function of the rate of return on the assets of the firm $\tilde{r}_F$. As has been previously explained, this is because each time there is a change in $\eta_S$ (e.g., a change in $V$, $\sigma^2$, $r_F$ or $T$), the relationship between $\tilde{r}_F$ and $\tilde{r}_S$ changes. Consequently, even if the expected rate of return on the firm's assets $\tilde{r}_F$ is a stationary process, the variable $\tilde{r}_S$ will not follow a stationary process. Hence, the random walk assumption for common stock is at best a first approximation, and for a certain class of firms it is simply incorrect. This analysis is consistent with the empirical findings of Officer (1971) and the theoretical probability model of Press (1967). Officer found that common stock returns have a fat-tailed distribution (relative to a normal distribution) with a stable and finite variance which converges toward normality with additional observations. Such a random process is consistent with a non-stationary normal process, as shown by Press.

Our model has important implications for tests of the validity of the CAPM using returns data of levered equity. In Merton (1970) there is a warning about using equity returns in empirical studies:

Although the value of the firm follows a single dynamic process with constant parameters . . . the individual component securities follow a more complex process with changing expected returns and variances. Thus, in empirical examination using a regression . . ., if one were to use equity instead of firm values, systematic biases will be introduced.

Black-Jensen-Scholes (1972) and Fama-MacBeth (1973) have developed techniques for testing the CAPM which avoid selection bias due to the regression phenomena. Essentially, they estimate common stock $\beta$'s in one period and then use these estimates to test the CAPM on a later period of data. In addition, they aggregate individual securities which have non-stationary $\beta$'s into portfolios with more stationary $\beta$'s. These portfolios' $\beta$'s are then estimated over an average of nine years of monthly data, implying that the portfolio $\beta$'s are indeed stationary.

56 Also, there would be a rise in systematic risk and, therefore, in expected rate of return of both debt and equity due to the rise in leverage resulting from the dividend payments. See Case Study IV and substitute dividend payment for spin-off.

57 Fama (1970) rightfully pointed out that stationarity is not a necessary condition for the existence of an efficient capital market.

58 The generality of this conclusion depends on the assumption of stationarity for the firm's systematic risk. Also see footnote 17.

over that period. This should be tested, not assumed. But more importantly, the aggregation of non-stationary individual securities to obtain stationary portfolios of securities should be closely scrutinized to see if this is eliminating the problem or only obscuring it. One alternative statistical technique which shows promise is a random coefficient model which Rosenberg (1973), among others, has recently been studying.

Turning once again to the measurement of market risk, our analysis suggests that the proxy for the market index of asset returns should not consist entirely of equity. Such a market index can be expected to be an upward biased estimate of market risk.\(^6\) This, in turn, causes a downward bias in the estimates of individual asset's systematic risk.\(^7\) One would suspect that more stable estimates of assets' systematic risk could be obtained by using a market index including firm debt.

The conclusion from this discussion is that the statistical methodology generally used to estimate corporate securities' risk has much to be desired. The problems associated with non-stationary security return distributions have rarely been faced directly; this is especially important when major asset or capital structure changes occur in the period in which the firms are being studied. It is hoped that our analysis will provide some structure in the pursuit of better techniques of estimating market risk.

Appendix I

(A) Partial derivatives of the option pricing equation

Partial derivatives of the option pricing equation,

\[ S = VN(d_1) - C e^{-rt}N(d_2) > 0, \]

are as follows:

\[ S_V = N(d_1) > 0, \]
\[ S_C = -e^{-rt}N(d_2) < 0, \]
\[ S_{e^T} = C e^{-rt}Z(d_2) \frac{\sqrt{T}}{\sigma} > 0, \]

\(^6\)See footnote 16.

\(^7\)This can be clearly seen if we assume that there exists only one representative firm. Using eq. (3), we can show that the systematic risk of any asset \(i\) is

\[ \beta_{iV} = \frac{\text{cov} (r_i, \tilde{T})}{\sigma^2(r_i)} = \frac{n_s^2 \text{cov} (r_i, r_s)}{\eta_s \text{cov} (r_i, r_s)} \equiv \eta_s \beta_{iS}, \text{ where } \eta_s \geq 1. \]

\(\beta_{iS}\) is the measured systematic risk of asset \(i\) when the equity of our representative firm is used as a proxy for the entire firm.
\[ S_r = TC e^{-rT} N(d_2) > 0, \]
\[ S_T = C e^{-rT} \left[ Z(d_2) \frac{\sigma}{2 \sqrt{T}} + r_F N(d_2) \right] > 0, \]
where
\[ Z(d_1) = \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} = \text{the standard normal density at } d_1. \]

For the partial derivatives of debt see footnote 11.

(B) Partial derivatives of \( \beta_D \)

The partial derivatives of the systematic risk of debt where the firm’s systematic risk is stationary and positive, i.e., \( \partial \beta_D / \partial K = 0 \) and \( \beta_D > 0 \) are as follows:

\[
R \equiv \frac{VN(-d_1) C e^{-rT} N(d_2)}{D^2 \sigma \sqrt{T}},
\]

\[
\frac{\partial \beta_D}{\partial V} = -\frac{R}{V} \left[ \frac{Z(-d_1)}{N(-d_1)} + \frac{Z(d_2)}{N(d_2)} - \sigma \sqrt{T} \right] \beta_D < 0,
\]

\[
\frac{\partial \beta_D}{\partial C} = \frac{R}{C} \left[ \frac{Z(-d_1)}{N(-d_1)} + \frac{Z(d_2)}{N(d_2)} - \sigma \sqrt{T} \right] \beta_D > 0,
\]

\[
\frac{\partial \beta_D}{\partial r_F} = -\frac{RT}{N(-d_1)} \left[ \frac{Z(-d_1)}{N(-d_1)} + \frac{Z(d_2)}{N(d_2)} - \sigma \sqrt{T} \right] \beta_D < 0,
\]

\[
\frac{\partial \beta_D}{\partial \sigma^2} = \frac{R \sqrt{T}}{2 \sigma} \left[ \frac{d_2}{N(-d_1)} + d_1 \frac{Z(d_2)}{N(d_2)} \right] \beta_D \equiv 0,
\]

\[
\frac{\partial \beta_D}{\partial T} = R \left[ -r_F \left( \frac{Z(-d_1)}{N(-d_1)} + \frac{Z(d_2)}{N(d_2)} - \sigma \sqrt{T} \right) \right.
\]
\[
+ \frac{\sigma}{2 \sqrt{T}} \left( d_2 \frac{Z(-d_1)}{N(-d_1)} + d_1 \frac{Z(d_2)}{N(d_2)} \right) \beta_D \equiv 0,
\]

where

\[ \frac{Z(-d_1)}{N(-d_1)} + \frac{Z(d_2)}{N(d_2)} - \sigma \sqrt{T} > 0, \]

\[ ^{*2} \text{We also assume that the firm is composed of physical assets, riskless debt or unlevered equity but not other financial assets. See footnote 18.} \]
\[ \frac{d_2 Z(-d_1)}{N(-d_1)} + \frac{d_1 Z(d_2)}{N(d_2)} = 0, \]
and \( Z(d_1) = Z(-d_1). \)

(C) Partial derivatives of \( \beta_S \)

The partial derivatives of the systematic risk of a call option where the underlying asset's systematic risk is stationary and positive, i.e., \( \frac{\partial \beta_v}{\partial K} = 0 \) and \( \beta_v > 0 \) are as follows:

\[ Q = \frac{VN(d_1) C e^{-rT} N(d_2)}{S^2 \sigma \sqrt{T}} > 0, \]

\[ \frac{\partial \beta_S}{\partial V} = -Q \left[ \frac{Z(d_1)}{N(d_1)} - \frac{Z(d_2)}{N(d_2)} + \sigma \sqrt{T} \right] \beta_v < 0, \]

\[ \frac{\partial \beta_S}{\partial C} = Q \left[ \frac{Z(d_1)}{N(d_1)} - \frac{Z(d_2)}{N(d_2)} + \sigma \sqrt{T} \right] \beta_v > 0, \]

\[ \frac{\partial \beta_S}{\partial r_F} = -QT \left[ \frac{Z(d_1)}{N(d_1)} - \frac{Z(d_2)}{N(d_2)} + \sigma \sqrt{T} \right] \beta_v < 0, \]

\[ \frac{\partial \beta_S}{\partial \sigma^2} = -Q \sqrt{T} \left[ d_1 \frac{Z(d_1)}{N(d_1)} - d_2 \frac{Z(d_1)}{N(d_1)} \right] \beta_v < 0, \]

\[ \frac{\partial \beta_S}{\partial T} = -Q \left[ r_F \left( \frac{Z(d_1)}{N(d_1)} - \frac{Z(d_2)}{N(d_2)} + \sigma \sqrt{T} \right) \right. \]
\[ + \left. \frac{\sigma}{2 \sqrt{T}} \left( d_1 \frac{Z(d_1)}{N(d_2)} - d_2 \frac{Z(d_1)}{N(d_1)} \right) \right] \beta_v < 0, \]

where

(C) \( d_1 \frac{Z(d_1)}{N(d_1)} - d_2 \frac{Z(d_1)}{N(d_1)} > 0, \)

(D) \( \frac{Z(d_1)}{N(d_1)} - \frac{Z(d_2)}{N(d_2)} + \sigma \sqrt{T} > 0, \)

and \( VZ(d_1) = C e^{-rT} Z(d_2). \)

63 See preceding footnote.
(D) Proofs of the inequalities

Using the upper bound of the Mills ratio,\(^6^4\) we can show that

\[ \frac{Z(d)}{N(d)} > -d, \quad \text{for} \quad -\infty < d < \infty. \]

From this inequality we can see that

\[ \frac{Z(-d_1)}{N(-d_1)} + \frac{Z(d_2)}{N(d_2)} - \sigma \sqrt{T} > d_1 - d_2 - \sigma \sqrt{T} = 0. \]

Transforming eq. (B) we see that

\[ d_2 \frac{Z(-d_1)}{N(-d_1)} + d_1 \frac{Z(d_2)}{N(d_2)} = [d_2 C e^{-rT} N(d_2) + d_1 VN(-d_1)] \]

\[ \times [C e^{-rT} N(d_2)(Z(-d_1))^{-1} N(-d_1)]^{-1} \]

\[ = [d_1 D - \sigma \sqrt{T} C e^{-rT} N(d_2)] \]

\[ \times [C e^{-rT} N(d_2)(Z(-d_1))^{-1} N(-d_1)]^{-1} \leq 0, \]

since

\[ d_1 \geq \sigma \sqrt{T} \left( \frac{C e^{-rT} N(d_2)}{D} \right), \quad \text{where} \quad 0 \leq \left( \frac{C e^{-rT} N(d_2)}{D} \right) \leq 1. \]

So expression (B) is always greater than zero, for \( d_1 > \sigma \sqrt{T} \) or \( V > C e^{-\left( r + \frac{1}{2} \sigma^2 \right)T} \).

Transforming eq. (C) we find

\[ d_1 \frac{Z(d_2)}{N(d_2)} - d_2 \frac{Z(d_1)}{N(d_1)} = [d_1 VN(d_1) - d_2 C e^{-rT} N(d_2)] \]

\[ \times [C e^{-rT} N(d_2)(Z(d_1))^{-1} N(d_1)]^{-1} \]

\[ = [d_1 S + \sigma \sqrt{T} C e^{-rT} N(d_2)] \]

\[ \times [C e^{-rT} N(d_2)(Z(d_1))^{-1} N(d_1)]^{-1}. \]

\(^6^4\)See Gordon's (1941) upper bound on the Mill's ratio, \( N(-t)/Z(-t) \) for \( t > 0 \).
This will be positive if
\[ d_1 > -\sigma \sqrt{T} \left[ \frac{C e^{-rT} N(d_2)}{S} \right], \]
where \( d_2 \) is defined later. Therefore, eq. (C) will always be positive for firms where \( V \geq C e^{-(r+{k_1}^2)T} \), the firm's asset value at least equals the discounted face value of its debt; or, equivalently, when \( d_1 \geq 0 \). The only exception is the case where the firm experiences extreme losses causing \( V \leq C e^{-(r+{k_1}^2)T} \), where \( k \equiv (C e^{-rT} N(d_2))/S \).

Defining
\[ h(d) \equiv \frac{Z(d)}{N(d)} + d, \]
we know that \( h(d) \) is always positive from (E). Furthermore, it can be shown that \( h'(d) \geq 0 \) for all \( d \), which means that \( h(d) \) is a monotone strictly increasing function of \( d \). Now \( d_1 > d_2 \), so \( h(d_1) - h(d_2) > 0 \).

\[ \frac{Z(d_1)}{N(d_1)} - \frac{Z(d_2)}{N(d_2)} + \sigma \sqrt{T} \left[ \frac{Z(d_1)}{N(d_1)} + d_1 \right] - \left( \frac{Z(d_2)}{N(d_2)} + d_2 \right) \]
\[ = h(d_1) - h(d_2) > 0. \]

The first equality is based on the definition of \( d_2 \) at the beginning of section 4.

Appendix II

Numerical examples of the case of conglomerate merger

In our analysis of Case III, we showed how the value of equity of the merged firm \( G \) (denoted by \( S^G_0 \)) can be derived by using eq. (4). From a merger of \( A \) and \( B \), when \( C_G = C_A + C_B \), the equityholders will suffer a loss of \( L_S = S^A_0 + S^B_0 - S^G_0 \). Their position can be restored by increasing the face value of debt \( C_G \) to \( C_G' \) (where primes denote the firm with the new capital structure) so that \( D^G_0 (C_G'/C_G) - D^G_0 \) is equal to \( L_S \).

For example, assume
\[ V^A_0 = V^B_0 = \$$1000, \]
\[ S^A_0 = S^B_0, \]
\[ \sigma^2_A = \sigma^2_B = \sigma^2. \]

\( ^{b,5} \) A more detailed proof will be supplied by the authors upon request.
\[ C_A = C_B = \$500, \]

\[ T = 5 \quad \text{(e.g., 5 years)}, \]

\[ r = 0.08. \]

If \( \sigma^2 = 0.10 \), then

\[ S^A_0 = S^B_0 = \$675.2 \quad \text{and} \quad S^A_0 + S^B_0 = \$1350.4. \]

If the correlation between the percentage return on \( A \) and \( B \) is \( \rho = 0 \), then for the merged firm \( (C_G = C_A + C_B = \$1000) \),

\[ S^G_0 = \$1332.5 \quad \text{and} \quad D^G_0 = \$667.5, \]

and hence

\[ L^G = \$1350.4 - \$1332.5 = \$17.9. \]

If we issue additional debt with face value of \$560 and with the proceeds retire part of the equity\(^6\) we obtain

\[ S^{G'}_0 = \$1013.27 \quad \text{and} \quad D^{G'}_0 = \$986.7. \]

The market value of the old bonds is \$986.7 (1000/1560) = \$649.6, exactly like their combined value before the merger (i.e., \( D^A_0 + D^B_0 = 2 \times 324.8 = \$649.6 \)).

The wealth of the equityholders is now composed of the current market value of equity (\$1013.3) plus the amount of cash they have received, which is equal to the market value of the new debt (\$337.1), together totaling \$1350.4. By the merger, the 'debt capacity' of the firm has increased by approximately 50 percent.\(^6\) The following table gives the amount by which \( C_G \) should increase, in order to restore previous values, for a few values of \( \sigma^2 \) and \( \rho \).

<table>
<thead>
<tr>
<th>( \sigma^2 )</th>
<th>( \rho = 0.0 )</th>
<th>( \rho = 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>48.0%</td>
<td>20.0%</td>
</tr>
<tr>
<td>0.10</td>
<td>56.0%</td>
<td>23.6%</td>
</tr>
</tbody>
</table>

\(^6\)Before the debt is issued it is announced that an equal dollar amount of equity will be retired.

\(^6\)It should be noted that in an economy with perfect capital markets, where securityholders have complete protection against deterioration of their positions, the debt capacity of the firm is not an operative term, as the firm is indifferent to its capital structure [Modigliani-Miller (1958)]. Also refer to footnote 5.
References


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