Referral Alliance and Consumer Search*

Yat Fung Wong†

November 29, 2014

Abstract

In this paper we pursue three main objectives: (1) to develop a model of firm referrals in the presence of consumer search, (2) to investigate the market impacts of firms’ proactive engagement in referral alliances, (3) to identify the incentives for firms to enter a referral alliance. To achieve these purposes, we extend the Wolinsky’s (1986) model to a three-stage game with two types of products produced by a continuum of firms, with each one producing a single type only. In the first stage, firms simultaneously decide on the formation of referral alliances, in which each alliance consists of a pair of firms producing different types of products. In the second stage, they set price simultaneously. In the third stage, each consumer who only values one type of product searches sequentially for the right product. We show that firms with referral partner always earn higher profits than those without. Moreover, the proliferation of referral alliances always benefits consumers and society but not necessarily firms. On the one hand, the spike of referral alliances would induce a price reduction by encouraging consumers to search more. On the other hand, it increases the mass of consumers participating in the search market. Firms experience profit increases by building up the referral alliances either when they are the few one manage to do so or when referrals are able to attract a significant amount of new consumers to search in the market. With uniformly distributed willingness to pay, we show that firms are more likely to benefit from the introduction of referral alliances for sufficiently large search cost, sufficiently uncertain product value, or sufficiently high utility from an outside option.

*The author is deeply indebted to Hideo Konishi for his guidance, inspiration and encouragement. He is grateful to Utku Ünver and Susanto Basu for valuable suggestions. The author benefited from the discussion with Jinghan Cai, Chiu Yu Ko, and participants of the dissertation workshop in Department of Economics, Boston College. Needless to say, all errors are mine.
†Department of Economics, Boston College. Email: yat.wong@bc.edu
1 Introduction

A common challenge for all professional service firms, including law firms, accounting firms, advisory consulting firms in attracting new clients is that it is difficult for consumers to determine which firm is the best fit before visiting them. Additionally, it is often observed that consumers still might not fully understand the products even after costly search process due to lack of some specific knowledge. In view of this challenge, another common practice is that firms direct a visited consumer to another firm without involving fees when they do not have the expertise to deal with the problem.\(^1\) Since fees are not involved in referrals, it is natural that different types of firms might consider cooperating together to build up an alliance, in which they mutually agree to exchange their "residual demand". Nowadays, this kind of reciprocal relationship has been a ubiquitous phenomenon. According to Wikipedia, the number of networks in law firms based upon published directories found in Martindale Hubbell, HG.org, and Chambers and Partners has been growing rapidly and doubled over the past ten years. In addition, International Bar Association states that there were at least 134 prominent international law firm alliances in the world by October 2012.\(^2\) Moreover, it is not immediately clear what the market impacts are as more firms manage to engage in the alliances.

This paper intends to analyze the impacts of proliferating referral alliances on market outcomes and identify the incentives for firms to build up a referral alliance. To achieve these purposes, we develop a sequential search model by modifying Wolinsky (1986) to incorporate two types of products A and B. Each firm produces either product A or B, but consumers cannot identify which firms produce which product. Each consumer needs either product type A or B and the other type of product is useless for her. Consumers have differentiated tastes, and products are search goods: each consumer realizes her willingness to pay for the product sold by a firm, just after she visits the firm. Consumers visit firms sequentially if they decide to search in the market. In addition, we add to the consumer search model by allowing the possibility of referral alliance formation between two different types of firms. Each

\(^1\)Wheiler (1987) argues that referrals are important for professional service providers because of consumers' difficulties in problem recognition, causing a high level of uncertainty in product values and a high level of perceived risk in choosing a wrong firm. Furthermore, he contends that referrals facilitate market exchange by reducing the uncertainty in decisions.

\(^2\)An article appeared in "Law Firm Management News October 2012" concerning law firm alliances. For more information on the tendency for the spike of national and international referral alliances during this information age, one might refer to the book "Managing the Modern Law Firm" edited by Empson (2007).
firm first chooses whether to build up a referral alliance, in which each alliance consists of a pair of firms producing different types of products. The alliance is formed when they mutually agree to refer its unmatched consumers to the other one. After that, they make the price decision simultaneously. Lastly, consumers decide on the search and buying decision.

With the specified model, we show that an immediate effect of introducing a referral partner is that it increases effective demand for the two firms by twice. As a result, firms with the referral partner always earn higher profits than those without, so the unique equilibrium involves all firms being in the referral alliance. Moreover, the proliferation of referral alliances always benefits consumers and society but not necessarily firms. On the one hand, the spike of referral alliances would induce a price reduction by encouraging consumers to conduct more search. On the other hand, it increases the mass of consumers participating in the search market. Firms experience profit increases from the formation of more referral alliances either when they are the few one manage to do so or when referrals are able to attract a significant amount of new consumers to search in the market. Since all firms would have a referral partner in the market equilibrium, they are better off in the equilibrium only if consumers’ participation in the search market with referrals increases significantly. Assuming uniformly distributed willingness to pay, we show that firms are more likely to benefit from the emergence of more referral alliances for sufficiently large search cost, sufficiently uncertain product value, or sufficiently high utility from an outside option. Since there are high search costs and high levels of uncertainties in evaluating the products for professional service as it is argued in Wheiler (1987), our paper rationalize the phenomenon that referral alliances are more frequently observed in the markets with professional service providers like law and accounting firms.

One extension of the model is to interpret the outside option as a result of the existence of a large firm, which carries more products and has higher abilities in committing to provide certain utility to consumers. Armstrong and Vickers (2001) introduce a competition in utility space model by assuming firms’ profits to be a function of average utility per consumer. Shelegia and Wilson (2014) subsequently employ this concept to analyze price discrimination in sales markets. Under our scenario, we might assume that the large firm competes with small firms by supplying utility directly to consumers. When the large firm successfully commits to provide more utilities to consumers, less consumers are willingness to search in the market. Thus, the stronger large firm could provide small firms higher incentives to build up referral alliances so as to increase the amount of consumers participating in the search.
market. Therefore, our result is consistent with the observation that it is becoming more popular for small law firms or accounting firms from different jurisdictions joining together to build up referral alliances to compete with the nationalized or internationalized large corporations.\textsuperscript{3}

Another extension of the model is to introduce a change in consumers’ perception on the product quality upon receiving a referral. In our basic model, we assume that the only function for referrals is to direct the unmatched consumers to the partner firm. Since firms understand consumers’ problems better than consumers, referrals might also be used as an opportunity to enhance consumers’ trust in the suitability of the product provided by the partner firm. Peter Sandeen, a business consultant, argues that trust can be transferred forward, so that an effective referral might be used to create expert status and improve trust on the product quality.\textsuperscript{4} It is then natural to think that firms could benefit more from the additional function of referrals. Moreover, we show that firms’ profit remains the same as long as all firms are homogeneous and consumers search randomly. The higher perceived product quality just encourages consumers to search for firms even more actively, contributing to the same price at the equilibrium. When only a small portion of firms manage to join a referral alliance, firms inside the referral alliance could charge a higher price and enjoy a higher profit level. Thus, our previous results remain valid even if referrals are able to improve the perceived product value.

Referrals in the market are proliferating rapidly and have raised much discussion. Yet there have been limited works on the issue in economics. Spurr (1990) empirically shows that referrals between lawyers depend on various factors including the nature of the problem, the value and the advertising activities. Garicano and Santos (2004) examine referrals between two vertically differentiated firms and suggest that there are both adverse selection and moral hazard problems retarding efficient referrals, because of low quality firms’ incentive to keep high skill tasks and unobservable efforts from high skill firms. They argue that the formation of partnership is a good way to solve the problems. Moreover, their model does not include consumer search as an element. Arbatskaya and Konishi (2012) analyze referrals in a market

\textsuperscript{3}An article appeared in New York Times on June 8, 2001, “Making a network of lawyers; small firms find way to compete with giants” by Jonathan D.Glater, expresses the view that having referrals improve the samll firm’s ability in competing with the large firms.

Another article appeared in Chicago Daily Law Bulletin April 5, 2012, "Local lawyer leads global referral network" by John Flynn Rooney expresses similar view on forming of referral alliance.

\textsuperscript{4}"Give Referrals to Get Referrals: The Best Leads You’ve Ever Had" - an article appeared in Firepeople Marketing on January 13, 2012.
with horizontally differentiated firms and sequential consumer search. They work on a Salop circle model and define referral as a practice, in which a firm directs a consumer to an ideal product when the firm knows that she would not purchase from it. Their results show that the referral practice improves consumer match and weakens rival firms’ incentive to lower price, by comparing equilibrium outcomes when all firms make referrals with those when no firms make referrals.

In contrast, we endogenize referral alliance formation and analyze its impact on market outcomes in an environment with consumer search. Furthermore, we work on a different type of referral. Arbatskaya and Konishi (2012) consider referrals within the same type of products, while we consider referrals across type. Under our definition, referral occurs when a firm directs its unmatched consumers to visit its partner firm, but the referred consumers might still search in the market if they do not like the product in the partner firm. In other words, referrals in our model only make the partner firm prominent, without restricting the future search decision. In this sense, our paper is also related to consumer search with prominent firm literature.

Armstrong, Vickers and Zhou (2009) and Zhou (2009) analyze a search market with ex-ante prominent firms, which all consumers visit first. They show that a prominent firm charges a lower price because it has relatively more first visit consumers compared to the returning consumers, who are less sensitive to price increases. In our model, firms are ex-ante homogeneous to consumers because they do not publicly announce the referral policy. As a result, consumers have to visit firms randomly and receive referral only when they casually reach an unmatched firm in a referral alliance. Thus, firms in our model become prominent stochastically and consumers always expect it to happen with a same probability along the searching process. The possibility of getting referrals would increase the expected benefits from continuing to search, which enhances consumers’ incentive to visit more firms. This induces a lower market price in the referral equilibrium compared to the no referral environment.

In terms of modeling, the closest papers to ours are Bar-Isaac et al. (2012), Larson (2013) and Yang (2013). We all work on a continuum version of Wolinsky’s (1986) model by allowing firms to have additional choice other than price. While the three papers in the literature endogenize the choice of product design, we endogenize the choice of referral alliance formation.\(^5\) Even though referral alliances in our paper

\(^5\)Both Bar-Isaac et al. (2012) and Larson (2013) assume consumers value both types of the products but consumers’ perception on product value distribution would change when firms choose a different product design. In contrast, Yang (2013) assumes that there are multiple groups of consumers in which each group only value one type of the product and there is a change in product
could also serve the function of product design by altering perceived product quality, they differ from each other significantly. The forming of the referral alliance has an additional effect to increase firms’ effective demand by twice due to the reciprocal exchange of the unmatched consumers.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 considers the determinants of consumer search equilibrium and firm optimal price. In section 4, we analyze the market equilibrium, by assuming that the only function for referral is to direct the unmatched consumers to the partner firm. After that, we impose more structures by assuming uniformly distributed consumers and analyze the impacts of referral alliance formation at different levels of search cost. Section 5 involves some discussion on two extensions of the basic model. Section 6 contains a short conclusion.

2 The Model

Our search model follows Wolinsky (1986) with two modifications: a continuum of firms and two types of products. There is a continuum of risk neutral firms and consumers of measure 2 and 2n, respectively. There are two types of products and consumers. Each firm specializes in producing one type with zero marginal cost. Each consumer only demands one type of product and considers the other type to be of no value. We denote the types of product (consumer) by A and B (a and b), in which a type a (b) consumer derives positive value from type A (B) product only. When the consumer derives positive value from the firm, we say they match with each other. Before visiting a firm, each consumer believes that she matches with the firm with probability \( \frac{1}{2} \). In addition, when a consumer \( i \) matches with a firm \( j \), the consumption value is \( u + \varepsilon_{ij} \) where \( \varepsilon_{ij} \) follows an identical and independent distribution \( G(\varepsilon) \) over \( [\bar{\varepsilon}, \underline{\varepsilon}] \) such that \( 1 - G(\varepsilon) \) is strictly logconcave. Since both types of consumers behave in the same way and care about the product from a matched firm only, the two types of firms are symmetric in their price decision. We simply denote the price in firm \( j \) as \( p_j \) without mentioning the product type. The consumer \( i \) derives utility \( u + \varepsilon_{ij} - p_j \) from purchasing in the matched firm \( j \).

Each consumer could choose to search in the market or simply enjoy utility \( \tau \) from design when firms decide to produce a different type of the product to serve different group of consumers.
an outside option. We assume that each consumer observes her own outside option and believes that \( \tau \) follows a distribution \( H(\tau) \) over \( [\underline{\tau}, \overline{\tau}] \). If consumers participate in the search market, they will visit firms randomly and sequentially. A search cost \( s \) is needed for visiting each firm. The consumer has perfect recall and could buy from the previous visited firm without an additional search cost. The net utility of consumer \( i \) is \( u + \varepsilon_{ij} - p_j - ks \) if she buys the matched product from firm \( j \) after \( k \) visits.

We add to the search model with the possibility of referral alliance formation. Two different types of firms could form an alliance, by committing to refer each other their unmatched consumers without cost. Referral occurs when an unmatched consumer is directed to visit the partner firm, without involving an additional search cost and without any restriction on the buying decision.

The game has three stages. In the first stage, firms simultaneously decide on the formation of referral alliances. In the second stage, firms make their price decisions simultaneously. In the third stage, consumers choose the searching and buying decisions.

\section{Consumer Search and Market Price}

\subsection{Consumer Search Behavior}

In this subsection, we will argue that the consumer search equilibrium is governed by a stationary stopping rule. Let \( \tilde{p}^R \) and \( \tilde{p}^{NR} \) be the price expectation for firms with and without a referral partner, respectively, and \( \lambda \) be the proportion of firms inside a referral alliance.

Suppose \( v' \) be the current maximum achievable utility and a consumer decides to have an additional search. The consumer gains from this additional search only when she is able to derive higher utility from the newly visited firm. Otherwise, she does not gain from the additional search because she would go back to buy from the previous visited firm. With probability \( \frac{1}{2} (1 - \lambda) \) the consumer reaches a matched firm without the referral partner, she would gain from this search provided that the realized utility satisfies \( u + \varepsilon - \tilde{p}^{NR} > v' \). With probability \( \lambda \), the consumer reaches a firm inside the referral alliance so that she is able to find a matched product with
probability one, but she would buy the product only when \( u + \varepsilon - \tilde{p}^R > v' \) holds. Thus, the expected net gains from an additional search is

\[
\frac{1}{2} \left( 1 - \lambda \right) \int_{\tilde{p}^N - u}^{\varepsilon} (u + \varepsilon - \tilde{p}^R - v') \, dG(\varepsilon) + \lambda \int_{\tilde{p}^R - u}^{\varepsilon} (u + \varepsilon - \tilde{p}^R - v') \, dG(\varepsilon) - s.
\]

Define an utility threshold \( w \) such that

\[
s = \frac{1}{2} \left( 1 - \lambda \right) \int_{w + \tilde{p}^N - u}^{\varepsilon} (u + \varepsilon - \tilde{p}^N - w) \, dG(\varepsilon)
\]

\[+ \lambda \int_{w + \tilde{p} - u}^{\varepsilon} (u + \varepsilon - \tilde{p} - w) \, dG(\varepsilon) \tag{1}
\]

Since \( U(\varepsilon - U) \, dG(\varepsilon) \) is strictly decreasing in \( U \), there is an unique \( w \) satisfying equation (1). We will show that a consumer’s search behavior is governed by the utility threshold \( w \) and her outside option \( \tau \). To carry on the analysis, we will first work on an environment without outside options and obtain our Lemma 1, Lemma 2 and proposition 1. After that, we will argue that the results could be extended to the environment with heterogeneous outside options for consumers easily.

**Lemma 1.** Suppose \( w < 0 \). No consumers are willingness to search in the market.

**Proof.** The expected net gains from participating in the search market is

\[
\frac{1}{2} \left( 1 - \lambda \right) \int_{\tilde{p}^N - u}^{\varepsilon} (u + \varepsilon - \tilde{p}^N) \, dG(\varepsilon) + \lambda \int_{\tilde{p} - u}^{\varepsilon} (u + \varepsilon - \tilde{p}) \, dG(\varepsilon) - s = \frac{1}{2} \left( 1 - \lambda \right) \int_{\tilde{p}^N - u}^{\varepsilon} (u + \varepsilon - \tilde{p}^N) \, dG(\varepsilon) + \lambda \int_{\tilde{p} - u}^{\varepsilon} (u + \varepsilon - \tilde{p}) \, dG(\varepsilon) - \frac{1}{2} \left( 1 - \lambda \right) \int_{w + \tilde{p}^N - u}^{\varepsilon} (u + \varepsilon - \tilde{p}^N - w) \, dG(\varepsilon) - \lambda \int_{w + \tilde{p} - u}^{\varepsilon} (u + \varepsilon - \tilde{p} - w) \, dG(\varepsilon)
\]

Not that the following holds for any given \( p \).
\[
\int_{p-u}^{u} (u + \varepsilon - p) \, dG(\varepsilon) - \int_{w+p-v}^{\infty} (u + \varepsilon - p - w) \, dG(\varepsilon)
\]
\[
= \int_{p-u}^{w+p-u} (u + \varepsilon - p) \, dG(\varepsilon) + w \int_{w+p-u}^{\infty} \, dG(\varepsilon)
\]
\[
< 0 \text{ when } w < 0.
\]

This implies that the expected net gains from searching is negative whenever \( w < 0 \). \( \square \)

Lemma 1 shows that consumers will enter the search market if and only if \( w \) is non-negative. Next, we show in proposition 1 that consumers will stop searching and buy immediately as long as the realized utility from a firm exceed \( w \).

**Proposition 1.** Suppose \( w \geq 0 \). The consumer \( i \) will stop search and buy from firm \( j \) immediately if and only if \( u + \varepsilon_{ij} - p_{j} \geq w \).

**Proof.** Suppose not. Let firm \( j \) is the first sampled firm with \( \varepsilon_{ij} \geq w + p_{j} - u \) but the consumer decides to continue searching. Let \( v \) be the realized utility from firm \( j \). The expected net gain for an addition search is

\[
\frac{1}{2} (1 - \lambda) \int_{v + \tilde{p}^{NR} - u}^{\infty} (u + \varepsilon - \tilde{p}^{NR} - v) \, dG(\varepsilon) + \lambda \int_{v + \tilde{p}^{R} - u}^{\infty} (u + \varepsilon - \tilde{p}^{R} - v) \, dG(\varepsilon) - s.
\]

Since \( v \geq w \) and \( \int_{U}^{\infty} (\varepsilon - U) \, dG(\varepsilon) \) is strictly decreasing in \( U \), by equation (1) the expected net gains for an addition search is at most 0. This contradicts that the presumption that consumer decides to have an additional search. \( \square \)

Furthermore, Lemma 2 below shows that \( w \) equals the the expected net consumer surplus from searching. Denote \( \rho \) as the expected probability that a consumer buys from a visited firm. With proposition 1, we have the following equation for \( \rho \).

\[
\rho = \lambda \left[ 1 - G\left( \tilde{p}^{R} + w - u \right) \right] + \frac{1 - \lambda}{2} \left[ 1 - G\left( \tilde{p}^{NR} + w - u \right) \right]
\]

(2)

**Lemma 2.** The expected net consumer surplus for searching in the market is \( w \).
Proof. The expected gross consumer surplus and total search cost are

\[
E (u + \varepsilon - \tilde{p} \mid \varepsilon \geq w + \tilde{p} - u) = \frac{(1 - \lambda)}{2} \int_{\frac{w + \tilde{p}^{NR} - u}{\rho}}^{\tilde{p}} \frac{(u + \varepsilon - \tilde{p}^{NR})}{\rho} dG(\varepsilon) + \lambda \int_{\frac{w + p^{NR} - u}{\rho}}^{\tilde{p}} \frac{(u + \varepsilon - \tilde{p}^{R})}{\rho} dG(\varepsilon)
\]

and

\[
s \sum_{i=0}^{\infty} (1 - \rho)^i = \frac{s}{\rho}.
\]

Using equation (1), the expected net consumer surplus is

\[
E (u + \varepsilon - \tilde{p} \mid \varepsilon \geq w + \tilde{p} - u) - \frac{s}{\rho} = w
\]

With the above results, we are ready to show the consumers’ search behavior in the environment with heterogeneous outside options. Note that a consumer could never search in the market and earn the utility form her outside option at the same time. This immediately implies that consumers with different outside options would behave in the same ways as long as they decide to stay in the search market. In addition, the expected surplus for consumers to search in the market is independent of the search history because of the continuum of firms assumption. That means consumers will never switch their decisions between searching in the market and taking the outside option. Therefore, the expected consumer surplus in the search market always equals \(w\) independent of the outside options. As a result, the consumer with outside option \(w\) will search in the market if and only if \(w \geq \tau\) holds. In other words, consumers are sorted in taking the outside options and searching in the market according their outside options.

3.2 Firm Profit Maximization

Proposition 1 shows that a visited consumer buys the product from firm \(j\) provided that \(u + \varepsilon - p_j \geq w\). Given that the ex-ante probability for the consumer reaching
a matched firm is only $\frac{1}{2}$, the expected demand per unit of visited consumer is $\frac{1}{2} \left[ 1 - G (p_j + w - u) \right]$. Additionally, an extra demand $\frac{1}{2} \left[ 1 - G (p_j + w - u) \right]$ could be generated if the firm is in a referral alliance, due to the referrals from the partner firm. Thus, the expected demand per unit of consumer for firms with and without referral partner are $\left[ 1 - G (p_j + w - u) \right]$ and $\frac{1}{2} \left[ 1 - G (p_j + w - u) \right]$, respectively.

Expecting the consumer surplus for participating in search market to be $w$, a consumer with outside option $\tau$ will search in the market if and only if $w \geq \tau$. The expected consumer participation rate in the search market is then equal to $H (w)$. In addition, the effective total demand in the market is calculated as $2n \sum_{i=0}^{\infty} (1 - \rho)^i$ or $\frac{2n}{\rho}$. Thus, the expected profits for firms with and without the partner are, respectively

\[
\begin{align*}
\Pi^R &= \frac{2n}{\rho} H (w) p_j \left[ 1 - G (p_j + w - u) \right] \\
\Pi^{NR} &= \frac{n}{\rho} H (w) p_j \left[ 1 - G (p_j + w - u) \right]
\end{align*}
\]

Since $n$, $\rho$ and $H (w)$ are all independent of the price decision for an individual firm, each firm’s objective is to choose price $p_j$ to maximize the per consumer profit $p_j \left[ 1 - G (p_j + w - u) \right]$. The assumption of $1 - G (\varepsilon)$ being strictly logconcave ensures that the per consumer profit is strictly quasiconcave. Thus, the first order condition implies a unique profit maximizer $p$ determined by equation (3) below.

\[
p = \frac{1 - G (p + w - u)}{g (p + w - u)} \quad (3)
\]

The equation (3) indicates that firms charge for the same price independent of whether they are inside the referral alliance or not, and the optimal price is constrained by the consumer search behavior. Lemma 3 below shows that the firm price has to be lower, when the consumers become more demanding in choosing the product. Furthermore, the firm price is inelastic to the change in search behavior so that the price reduction is smaller than the increase in the threshold utility. Thus, both the consumers and society will benefit as the consumers become more demanding in the buying decision.

**Lemma 3.** The optimal price $p$ is strictly decreasing with $w$ and the social welfare $p + w$ is strictly increasing in $w$.  

10
\textbf{Proof.} Suppose $p$ is increasing in $w$. Then the increase in $w$ will make $p$ to remain unchanged or be greater, which implies that $p + w$ becomes larger. Moreover, it implies that the left side of equation (3) becomes smaller, because \( \frac{1 - G(\varepsilon)}{g(\varepsilon)} \) is strictly decreasing in $\varepsilon$ implied by the strictly logconcave of $1 - G(\varepsilon)$. This contradicts our presumption that $p$ is at least the value as before. Similarly, we can show that $p + w$ is strictly increasing in $w$. □

The analysis by now illustrates that our problem can be reduced to the determination of the threshold utility $w$. Denote $s^e = \frac{2s}{1+\lambda}$ as the effective search cost. Lemma 4 below shows that the consumers become more demanding in product selection whenever there is a reduction in the effective search cost.

\textbf{Lemma 4.} The threshold utility $w$ is strictly decreasing in $s^e$.

\textbf{Proof.} Since it is expected that all firms charge for the same price $p$, equation (1) for search equilibrium can be rewritten as

\[ \int_{w+p-w}^{\varepsilon} (u + \varepsilon - p - w) dG(\varepsilon) = \frac{2s}{1+\lambda} \quad (1') \]

Suppose on the contrary that $w$ is increasing in $s^e$. By Lemma 3, it must be that $p + w$ becomes larger as well. This implies that the left side of equation (1') becomes smaller, contradicting our presumption that $s^e$ increases. Thus, it must be that $w$ is strictly decreasing in $s^e$. □

A direct corollary of Lemma 4 is that the threshold utility $w$ is strictly increasing with $\lambda$, so consumers become better off as more referral alliances are formed. In addition, we will further show in proposition 2 that the spike of referral alliances also increases expected social welfare but it might decrease firm profits because of the lower market price. Moreover, two firms outside the alliance always want to cooperate together to build up a new alliance due to the increase in effective demand effect from the introduction of a referral partner. Let $\Pi^R(\lambda)$ and $\Pi^{NR}(\lambda)$ denote the firm profits with and without the referral partner in an environment with $\lambda$ proportion of firms having in the alliance. Furthermore, we denote $\Pi(0)$ and $\Pi(1)$ as the firms' profit in the environment with no firm and all firms introduced referrals, respectively.

\textbf{Proposition 2.} 1) The expected market price $p(\lambda)$ is continuous and decreasing.
in $\lambda$; 2) The expected social welfare $p(\lambda) + w(\lambda)$ is continuous and increasing in $\lambda$; 3) It must be that $\Pi^R(\lambda) > \Pi^{NR}(\lambda)$ for any $\lambda$ and $\Pi^R(\lambda) > \Pi(0)$ for sufficiently small $\lambda$.

**Proof.** First, the continuity of $p(\lambda)$ and $p(\lambda) + w(\lambda)$ follows by our assumption on well behavior probability function. Lemma 3 shows that $p$ is strictly decreasing with $w$ and $p + w$ is strictly increasing in $w$. But Lemma 4 implies that $w$ is strictly increasing in $\lambda$. Thus, it must be that $p$ and $p + w$ are strictly decreasing and strictly increasing with $\lambda$, respectively.

Second, with the same price in firms with and without referrals, the equation (2) can be rewritten as

$$
\rho = \frac{1+\lambda}{2} (1 - G(p + w - u)).
$$

So the expected firm profits with and without referrals can be rewritten as $\Pi^R(\lambda) = \frac{4n}{1+\lambda} H(w)p$ and $\Pi^{NR}(\lambda) = \frac{2n}{1+\lambda} H(w)p$, respectively. It implies that $\Pi^R(\lambda) > \Pi^{NR}(\lambda)$ for any $\lambda$.

Lastly, since $w(\lambda) > w(0)$ and $p(\lambda) < p(0)$ for all $\lambda > 0$, it must be that $H(w(\lambda)) > H(w(0))$ and $\frac{1}{1+\lambda}p(\lambda) < p(0)$. By continuity of $p(\lambda)$, we can choose $\lambda$ sufficiently small such that $\frac{1}{1+\lambda}p(\lambda)$ is smaller than $p(0)$ but $\frac{2}{1+\lambda}p(\lambda)$ is greater than $p(0)$. Given that $\Pi^R(\lambda) = \frac{4n}{1+\lambda} H(w(\lambda))p(\lambda)$ and $\Pi(0) = 2n H(w(0))p(0)$, it must be that $\Pi^R(\lambda) > \Pi(0)$ for sufficiently small $\lambda$. $\square$

The results in proposition 2 are quite intuitive. Referral makes the partner firm prominent by inducing the unmatched consumers to visit her first. As a result, it increases the effective demand of the partner firm by twice, contributing to a better performance of firms with referral partner than those without. Moreover, as more firms are in the referral alliances, consumers expect to gains more from an additional search, inducing them to be more demanding in choosing the product and search more actively. This intensifies market competition forcing down the market price, so that there is an increase in expected consumer surplus. Under the logconcave assumption the decrease in price is smaller than the increase in consumer surplus, causing a higher social welfare as well. With only a small portion of firms joining a referral alliance, they could earn higher profits comparing with the environment with no referrals because the effective demand for firms with referrals jumps up by twice
but the market price only decreases slightly. The remaining problem is whether firms manage to earn a higher profit in the market equilibrium.

4 Market Equilibrium

4.1 Referrals to Motivate Consumer Participation

We define the equilibrium in the first stage as a situation, such that there is no incentive for a small subgroup of firms acting together to change the decisions on the formation of referral alliances. A market equilibrium is then characterized by \( \{ \lambda^*, p_j^*, w^* \} \) such that

\[(i)\hspace{1em}\text{Given } \lambda^*, \text{ each consumer rationally expects the price } p_j^* \text{ in each firm, and the threshold utility } w^* \text{ is determined by consumers’ optimal search behavior determined by equation (1).}\]

\[(ii)\hspace{1em}\text{Given } \lambda^*, \text{ each firm } j \text{ rationally expects the threshold utility } w^*, \text{ and the firm price } p_j^* \text{ is determined by the firm’s profit maximization behavior from equation (3).}\]

\[(iii)\hspace{1em}\text{In the first stage, it must be that } \Pi_j(\lambda^*) \geq \Pi_j(\lambda^* \pm \delta) \text{ for all firm } j \text{ and for all sufficiently small } \delta > 0.\]

Since both \( p_j^* \) and \( w^* \) are uniquely determined by \( \lambda^* \), we denote the market equilibrium by \( \lambda^* \) in short. Our results show that all firms are inside a referral alliance in the market equilibrium, but they do not necessarily experience net gains from it.

**Proposition 3.** 1) The unique market equilibrium is \( \lambda^* = 1 \). 2) Firms’ profits could be higher or lower in the market equilibrium compared to the environment with no referrals.

**Proof.** 1) By proposition 2, it must be that \( \Pi^R(\lambda) > \Pi^{NR}(\lambda) \) for all \( \lambda \) and \( p(\lambda) \) is continuous and decreasing in \( \lambda \). For any \( \lambda < 1 \), we can choose \( \delta \) sufficiently small such that \( \frac{p(\lambda + \delta)}{\lambda + \delta} < \frac{p(\lambda)}{\lambda} \) slightly but \( \frac{4n}{1+\lambda+\delta} p(\lambda + \delta) > \frac{2n}{1+\lambda} p(\lambda) \). In addition, Lemma 4 shows that \( H(w) \) is increasing in \( \lambda \). That means \( \Pi^R(\lambda + \delta) > \Pi^{NR}(\lambda) \) holds for sufficiently small \( \delta > 0 \). Therefore, \( \lambda < 1 \) cannot be a market equilibrium. Similarly, we can show that \( \Pi(1) > \Pi^{NR}(1 - \delta) \) for all sufficiently small \( \delta > 0 \). Thus, the
unique market equilibrium is $\lambda^* = 1$.

2) The expected firm profits for $\lambda = 1$ and $\lambda = 0$ are $\Pi (1) = 2nH (w (1)) p (1)$ and $\Pi (0) = 2nH (w (0)) p (0)$, respectively. Since $p (1) < p (0)$ and $H (w (1)) > H (w (0))$, firms experience a profit increases only if the increase in $H (w)$ is high enough by introducing referrals. □

Intuitively, two firms always experience a jump up in profit by building up a referral alliance, because the formation of the referral alliance always increases their effective demand of by twice without additional cost and without affecting expected market price in a significant way. Thus, this creates a prison's dilemma problem, so that two different types of firms without referral partner always find it profitable to come together to build up a new alliance. Consequently, all firms will join the referral alliance in the market equilibrium. Moreover, the effective search cost in the market equilibrium would decrease from $2s$ to $s$, which motivates consumers to search more actively inducing a lower market price. Therefore, firms could be better off at the market equilibrium, only if the price reduction manages to attract a large amount of new consumers to search in the market.

4.2 Uniformly Distributed Consumers with Linear Demands

In this subsection, we impose more structures to the model so as to obtain more analytical results. We assume that the distributions for consumers’ outside option, $H (\tau)$, and their idiosyncratic preference on the matched product, $G (\varepsilon)$, are uniformly distributed so that there are uniformly distributed consumers with linear demands.

Proposition 4. Denote $\tilde{s} = \frac{\overline{(\tau-\tau+u)^2}}{4(2+\sqrt{2} (\tau-\varepsilon))}. \text{Firms benefit from the possibility of building up a referral alliance for all } s \geq \tilde{s}$.

Proof. Since the unique market equilibrium is $\lambda^* = 1$, we only need to compare firms’ profit for $\lambda = 0$ to that for $\lambda = 1$. It is easy to show that the market outcomes

\text{We assume that it is costless to build up a referral alliance. If we introduce cost of forming referral alliances, the market equilibrium remains at $\lambda^* = 1$ for sufficiently low cost and it becomes $\lambda^* = 0$ for sufficiently high cost. Additionally, there would be multiple equilibrium for intermediate levels of cost. Under uniform distribution, we manage to show that there are two equilibria form when the cost is in intermediate level.}
at $\lambda = 0$ and $\lambda = 1$ are, respectively

\[
\begin{align*}
\begin{cases}
p(0) &= 2\sqrt{(\bar{v} - \underline{v})s} \\
w(0) &= \bar{v} - 4\sqrt{(\bar{v} - \underline{v})s} + u \\
\Pi(0) &= 4n\frac{\bar{v} - 4\sqrt{(\bar{v} - \underline{v})s} + u}{\bar{v} - \underline{v}} \sqrt{(\bar{v} - \underline{v})s}
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
p(1) &= \sqrt{2(\bar{v} - \underline{v})s} \\
w(1) &= \bar{v} - 2\sqrt{2(\bar{v} - \underline{v})s} + u \\
\Pi(1) &= 2n\frac{\bar{v} - 2\sqrt{2(\bar{v} - \underline{v})s} + u}{\bar{v} - \underline{v}} \sqrt{2(\bar{v} - \underline{v})s}
\end{cases}
\end{align*}
\]

It is beneficial for all firms to introduce referrals if and only if

\[
2n\frac{\bar{v} - 2\sqrt{2(\bar{v} - \underline{v})s} + u}{\bar{v} - \underline{v}} \sqrt{2(\bar{v} - \underline{v})s} \geq 4n\frac{\bar{v} - 4\sqrt{(\bar{v} - \underline{v})s} + u}{\bar{v} - \underline{v}} \sqrt{(\bar{v} - \underline{v})s}
\]

\[
\Leftrightarrow s \geq \frac{(\bar{v} - \underline{v} + u)^2}{4(2 + \sqrt{2})^2(\bar{v} - \underline{v})}.
\]

Proposition 4 shows that firms experience profit increase for the possibility of building up referral alliance only in an environment with sufficiently large search cost. Additionally, the threshold search cost $s$ is strictly decreasing with $\bar{v} - \underline{v}$. Under uniform distribution, the variance of product valuation is strictly increasing with $\bar{v} - \underline{v}$. Therefore, our result implies that it is more likely for firms to earn higher profits by introducing the referral alliance when the consumers’ valuation on the firms’ products become more uncertain. To further analyze the profitability of firms in the enviroenmetn with proliferating referral alliances, we derive our proposition 5 on the changes in expected profits as the social norm of forming referral alliance develops.

**Proposition 5.** As the social norm of forming referral alliance develops, the change in profit for firms with a referral partner is as follows.
(1) For sufficient low search cost such that \(16 \sqrt{(\bar{\epsilon} - \underline{\epsilon}) s} < 3 (\bar{\epsilon} - \bar{\tau} + u)\), the optimal profit is decreasing with \(\lambda\).

(2) For sufficiently large search cost such that \(3 (\bar{\epsilon} - \bar{\tau} + u) < 8 \sqrt{2} \sqrt{(\bar{\epsilon} - \underline{\epsilon}) s}\), the optimal profit is increasing with \(\lambda\).

(3) For intermediate value of search cost such that \(8 \sqrt{2} \sqrt{(\bar{\epsilon} - \underline{\epsilon}) s} \leq 3 (\bar{\epsilon} - \bar{\tau} + u) \leq 16 \sqrt{(\bar{\epsilon} - \underline{\epsilon}) s}\), the optimal profit is first increasing and then decreasing with \(\lambda\).

**Proof.** The market outcomes for different \(\lambda\) are calculated as \(p(\lambda) = 2 \sqrt{\frac{(\bar{\epsilon} - \underline{\epsilon}) s}{1 + \lambda}}\) and \(w(\lambda) = \bar{\epsilon} - 4 \sqrt{\frac{(\bar{\epsilon} - \underline{\epsilon}) s}{1 + \lambda}} + u\). The optimal profit for firms with referrals is

\[
\Pi^R(\lambda) = \frac{8n}{\bar{\tau} - \bar{\tau}} \left[ (\bar{\epsilon} - \bar{\tau} + u) \sqrt{(\bar{\epsilon} - \underline{\epsilon}) s} \frac{(1 + \lambda)^{-3/2} - 4 (\bar{\epsilon} - \underline{\epsilon}) s}{(1 + \lambda)^{3}} \right]
\]

Differentiate the optimal profit function with respect to \(\lambda\).

\[
\frac{\partial \Pi^R(\lambda)}{\partial \lambda} = \frac{8n}{\bar{\tau} - \bar{\tau}} \left[ -\frac{3}{2} (\bar{\epsilon} - \bar{\tau} + u) \sqrt{(\bar{\epsilon} - \underline{\epsilon}) s} (1 + \lambda)^{-5} + 8 (\bar{\epsilon} - \underline{\epsilon}) s (1 + \lambda)^{-3} \right]
\]

\[
= 4n \sqrt{\frac{(\bar{\epsilon} - \underline{\epsilon}) s}{\bar{\tau} - \bar{\tau}}} (1 + \lambda)^{-3} \left( 16 \sqrt{(\bar{\epsilon} - \underline{\epsilon}) s} - 3 (\bar{\epsilon} - \bar{\tau} + u) \sqrt{1 + \lambda} \right)
\]

It is then easy to see the holding of proposition 5 □

Proposition 5 further illustrates that firms manage to increase profit by joining the referral alliance only if \(s\) and \(\bar{\epsilon} - \underline{\epsilon}\) are sufficiently large. If either \(s\) or \(\bar{\epsilon} - \underline{\epsilon}\) is small enough, the possibility of building up referral alliances enhances competition significantly making every firm worse off. Even though firms might become worse off in the market equilibrium, firms without referral partners still would like to enter a referral alliance because this could increase the effective demand without affecting market price in the short run. Moreover, firms might not want to compete by
forming referral alliance if they are all farsighted enough to expect that competing for introducing referrals might just make everyone worse off. With that in mind, our results could be used to rationalize the observation that referrals mainly exist in professional service industries where it is difficult to evaluate the product value and where there are infrequent purchase. While the uncertainty in assessment of the products is capture by a higher value of \( \bar{e} - \underline{e} \), the infrequent purchase of them contributes to a high search cost \( s \). Therefore, it is more likely for the professional service provider to enhance profit by introducing referral alliance as the mode of doing business.

5 Extensions

5.1 Referral as a Tool to Compete with a Large Firm

One extension of the model is to interpret the utility derived from the outside option as a result of the existence of a large firm, which usually carry more product and has higher abilities in committing to provide certain utility to consumers. Thus, we use Armstrong and Vickers’ (2001) competition in utility model to assume that the large firm competes with small firms by supplying utility directly to consumers. That means the large firm would choose the consumers’ utility to maximize her own profit. Denote \( c \) as the expected consumers’ surplus in visiting a large instead of searching in the market. Interpreting \( \tau \sim H (\tau) \) as the consumers’ heterogeneous preference between visiting a large firm and searching among small firms, the expected profits for small firms with and without the referral partner can be modified as

\[
\begin{align*}
\Pi^R (\lambda) &= \frac{4n}{1 + \lambda} H (w (\lambda) - c) p \\
\Pi^NR (\lambda) &= \frac{2n}{1 + \lambda} H (w (\lambda) - c) p
\end{align*}
\]

Under the modification, the increase in value of \( c \) will encourage more consumers to visit the large firm, so less of them are willingness to search in the market. Since we also show that the expected consumer surplus \( w (\lambda) \) is strictly increasing with \( \lambda \), small firms could increase consumers’ expected surplus from searching in the market by building up more referral alliances. That means the norm of entering referral alliances could help small firms to attract more consumers away from the large firm. Thus, we expect that there would be more referral alliances appeared in the economy,
as the large firm becomes more efficient in providing higher utilities to consumers. Our result then rationalize the statement that the formation of referral alliance could be employed as a tool to improve small firms’ ability in competing with the large firm. During the information age, the cost of forming and maintaining referral alliances become much lower and the large corporation become much more stronger. It is then expected that referral alliances will become more popular among small firms. This is consistent with the observation that there are growing number of national and international referral alliances formed by small law firms specializing in different areas.

5.2 Referral to Improve Perceived Product Quality

Our previous analysis assume that the only function of referrals is to direct the unmatched consumers to the partner firm. More realistically, since firms have much more information on the quality of the product than consumers, referrals might also serve as a means to improve consumers’ trust on the product from the partner firm. A natural question arises is whether firms can get more benefits if referral not only informs the consumer of the partner firm, but it also improves the consumers’ perception on the product quality. We will argue that as long as the consumers are not able to observe whether a firm is in the referral alliance ex-ante and all firms are homogeneous, the additional function for referrals will just alter the amount of consumers participating in the search market without affecting the market price. Because the increase in perceived product quality will motivate consumers to search in the market even more actively, so that firms can only charge for the same price as before.

Let $w^R > u$ denotes the perceived product quality for firms inside a referral alliance. Maintaining the random search assumption, the equations to determine search equilibrium and firms’ optimal price are modified as follows.

\[
s = \frac{1}{2} (1 - \lambda) \int_{w^R + p^{NR} - u}^{\infty} (u + \varepsilon - p^{NR} - w^R) dG(\varepsilon) \\
+ \lambda \int_{w^R + p^R - u^R}^{\infty} (u^R + \varepsilon - p^R - w^R) dG(\varepsilon)
\]
\[ p_j = \frac{1 - G \left( p_j + \overline{w}^R - u_j \right)}{g \left( p_j + \overline{w}^R - u_j \right)} \text{ for } u_j = u \text{ or } u^R \]

Firms’ profits with and without referrals become

\[
\begin{align*}
\Pi^R (\lambda) &= \frac{2n}{\bar{w}} H \left( \overline{w}^R \right) p^R \left[ 1 - G \left( p^R + \overline{w}^R - u^R \right) \right] \\
\Pi^{NR} (\lambda) &= \frac{n}{\bar{w}} H \left( \overline{w}^R \right) p^{NR} \left[ 1 - G \left( p^{NR} + \overline{w}^R - u \right) \right]
\end{align*}
\]

Firstly, by the logic of proposition 2 we can show that firms would benefit from having a referral partner when there is only a small portion of the firms do so. Secondly, we argue that it is still true that all firms would introduce referrals in the equilibrium. By the logic of proposition 3, it is sufficient to show that \( \Pi^R (\lambda) > \Pi^{NR} (\lambda) \) for any \( \lambda \), which can be shown by noting that firms with referrals always charge for a higher price than those without. Suppose on the contrary that \( p^R \leq p^{NR} \). It must be that \( p^R + \overline{w}^R - u^R < p^{NR} + \overline{w}^R - u \), which implies that \( p^R > p^{NR} \), contradicting our presumption. Thus, \( \lambda^* = 1 \) is still the unique market equilibrium, and the search equilibrium with and without improvement in perceived quality are as the follow.

\[
\int_{\overline{w} + p^R - u^R}^{\overline{w} + p^{R} - \overline{w}} \left( u^R + \varepsilon - p^R - \overline{w}^R \right) dG (\varepsilon) = s
\]

and

\[
\int_{\overline{w} + p - u}^{\overline{w} + p - \overline{w}} (u + \varepsilon - p - \overline{w}) dG (\varepsilon) = s.
\]

The two equations imply that \( \overline{w}^R + p^R - u^R = \overline{w} + p - u \) must hold in the equilibrium. Therefore, the additional function for the referrals to improve the perceived product quality does not change the equilibrium market price. That means the extra function of referrals would further increase the expected consumer surplus and the social welfare. Moreover, it is still true that firms might benefit from the introduction of referral alliance only when a sufficiently large amount of the new consumers are attracted to search in the market in the equilibrium with referrals.
6 Concluding Remarks

In this paper, we have developed a sequential search model with two types of firms and investigated the incentives for firms to build up a referral alliance. The formation of the referral alliance affects the market outcomes by altering the consumer search behavior in two aspects. On the one hand, it increases the effective demand for the firms in a referral alliance by inducing the unmatched consumers to visit their partner firm first. On the other hand, it enhances market competition and forces down market price by leading the consumers to be more demanding in the buying decision. By joining a referral alliance, firms would experience profit increase when there are only a small amount of firms do so. Moreover, firms inside the referral alliances always perform better than those outside because of our assumption on no costs of creating the alliance. As a result, firms always would like to enter the alliance whenever possible and the unique market equilibrium involves all firms introducing a referral partner. This would drive down the market prices, so firms could enjoy profits increase in the equilibrium only if the formation of referral alliances increases consumers’ incentive to search in the market significantly. Assuming uniformly distributed consumers’ willingness to pay, we also show that firms in referral alliances manage to increase profit for sufficiently large search cost, sufficiently uncertain product value or sufficiently large utility derived from an outside option. The results remain intact even if referrals could improve consumer’s trust in the product quality. Our analysis conveys the message that the possibility of referrals does not improve firm profit automatically. Giving referrals for the sake of giving referrals, by replicating the same strategy as the peers, might just benefit consumers without improving firm performance. Firms need to strive for something special so as to achieve a higher profit level by enjoying a comparative advantage in the market. There are two possible extensions that we leave for future research.

One question is whether firms in a referral alliance really invest more to increase the product quality. This kind of mechanism is missing in our current model. One way to analyze the problem is to consider the directed search for consumers. To incorporate this feature, we might assume that firms must invest some resource to build up the referral alliance and improve the product quality. Believing that the investment amount being positively correlated with the product quality, the consumers search in the alliance with higher investment amount first. If firms are homogeneous, it is expected that each alliance will have the same investment, and consumers have to search for the firms randomly. Similar arguments appeared in Bagwell and Ramey (1994) and Haan and Moraga-Gonzalez (2011) in showing the
effect of advertising on consumer search and firm profit. We conjecture that the same reasoning could be extended to analyze the effect of referral alliance as well.

Another question is whether firms have incentives to retain some of the unmatched consumers. In our model, they do not have incentive to do so because of our assumption that each firm is capable of producing one type of product only. More realistically, firms could have the ability to deal with multiple types of consumers, so they might mistreat some unmatched consumers to gain some rents instead of directing them to other firms as it is pointed out by Garicano and Santos (2004). With that in mind, an commitment to refer consumers truthfully would become important to determine whether the referral alliance could be sustained. Moreover, the analysis on this kind of mechanism is possible only after introducing multiproduct firms into the model as it is in Zhou (2014).
References


