In class, we showed that the arclength of the ellipse with equation
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a \geq b) \]
is given by the power series
\[ 2\pi a \left[ 1 - \sum_{k=1}^{\infty} \frac{P_k^2}{2k-1} \varepsilon^{2k} \right] \quad (1) \]
in powers of the eccentricity
\[ \varepsilon = \sqrt{1 - \frac{a^2}{b^2}}. \]

Some people use the simpler formula
\[ 2\pi \sqrt{\frac{a^2 + b^2}{2}}. \quad (2) \]

The goal of this problem is to use power series to show that the simple formula (2) is not the true arclength (1), but is a good approximation.

**Problem 1.** a) Show that
\[ 2\pi \sqrt{\frac{a^2 + b^2}{2}} = 2\pi a \sqrt{1 - \frac{\varepsilon^2}{2}} \]
(This is just algebra.)

b) Find a power series in \( \varepsilon \) for
\[ 2\pi a \sqrt{1 - \frac{\varepsilon^2}{2}} \]

c) Compare your power series in b) with the power series in (1). Why are they close? Which is bigger?

In class we have seen that the period \( T \) of a pendulum of length \( \ell \) is given by the integral
\[ T = 4\sqrt{\frac{\ell}{g}} \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1 - \sigma^2 \sin^2 \theta}}, \]
where \( \sigma = \sin(\theta_0/2) \) and \( \theta_0 \) is the initial angle of displacement.
Problem 2. a) Compute $T$ as a power series in $\sigma$.

b) Determine whether or not your series for $T$ converges when $\theta_0 = \pi$.

c) Physicists like to approximate the period by

$$T \approx 2\pi \sqrt{\frac{\ell}{g}}.$$ 

Explain why this is a good approximation only if $\theta_0$ is small.

---

Bessel functions arise in the study of vibrations of circular membranes and also in the orbits of planets. The first Bessel function is given by an integral

$$J_0(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \sin \theta) \, d\theta.$$ 

Problem 3. a) Compute a power series for $J_0(x)$ by starting with a power series for $\cos x$, replacing $x$ by $x \sin \theta$, and integrating term-by-term. What is the radius of convergence?

b) Check your answer in a) by computing the $n^{th}$ derivative $J_0^{(n)}(0)$ at $x = 0$. 