MATH 1103 Homework 6
Due Friday February 27, 2015

Problem 1. Find the volumes of the solids obtained by revolving the following graphs about the x-axis. Draw a picture of each solid.

a) \( y = 1 - x^2, \ -1 \leq x \leq 1 \)

b) \( y = \sin x, \ 0 \leq x \leq \pi \)

c) \( y = e^{-x}, \ 0 \leq x \leq b, \) and also \( 0 \leq x < \infty. \)

d) \( y = \frac{1}{x}, \ 1 \leq x \leq b, \) and also \( 1 \leq x < \infty. \)

Problem 2. Let \( a, b \) be positive constants. The equation

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

Defines an ellipse in the x, y plane.

a) Find the area of this ellipse. [Hint: Find the area of 1/4 of the ellipse, and use the substitution \( x = a \sin \theta. \)]

b) If we revolve this ellipse about the x-axis, we obtain a special kind of ellipsoid - one having circular cross-sections like a cigar. Find the volume of this ellipsoid.

Problem 3. Let \( a, b, c \) be positive constants. The equation

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
\]

Defines a general ellipsoid in three-dimensional x, y, z space. The z-coordinate varies between \(-c\) and \(c\). Each fixed value of \(z\) determines a slice of the ellipsoid, which is an ellipse with equation

\[
\frac{x^2}{a_z^2} + \frac{y^2}{b_z^2} = 1,
\]

where

\[
a_z = a \cdot \sqrt{1 - \frac{z^2}{c^2}}, \quad \text{and} \quad b_z = b \cdot \sqrt{1 - \frac{z^2}{c^2}}.
\]

Use Cavalieri’s principle and your result from part a) above to compute the volume of this ellipsoid. [The ellipsoid in part b) above is the special case \(c = b\).]

Problem 4. The goal of this problem is to compute the volume of a bagel. We will apply Cavalieri’s principle to the slices made by a bagel slicer. Our bagel is obtained by rotating the circle of radius \(a\) centered at \((0,b)\) about the x-axis, where \(a, b\) are constants with \(0 < a < b\). And \(x\) will vary from \(-a\) to \(a\).
a) What is the area of the bagel slice at \( x \)?

b) Integrate the above slices to obtain the volume of the bagel. (It is easier to find the volume of half a bagel, then multiply by two.)

**Problem 5.** Find the lengths of the following curves.

a) \( y = \frac{2}{3}(x - 1)^{3/2}, \ 1 \leq x \leq 2 \)

b) \( y = \log \cos x, \ 0 \leq x \leq \pi/4 \) (you will need \( \int \sec x \) which you can find on the back inside cover of our text; no need to derive it.)

c) \( y = x^2 - \frac{1}{8} \log x, \ 1 \leq x \leq 2. \)

d) \( y = x^{3/2}, \ 0 \leq x \leq 4. \)

This next problem relates the arc length integrand \( \sqrt{1 + \left( f'(x) \right)^2} \) to curvature. Don’t worry if you’ve never heard of curvature before; all you need to do this problem is right here. More discussion of curvature is found in my Math 1102 notes, section 3.6.

The *Curvature* of a graph \( y = f(x) \) is the function

\[
\kappa_f(x) = \frac{f''(x)}{(1 + (f'(x))^2)^{3/2}},
\]

which measures how much the curve is bending at the point \((x, f(x))\), where upward bending counts positive, and downward counts negative. The *total curvature* of the graph \( y = f(x) \) on \([a, b]\) is the integral of the curvature:

\[
\int_a^b \kappa_f(x) \, dx.
\]

Let \( \theta(x) \) be the angle of elevation made by the tangent line to the graph at \((x, f(x))\), in the positive \( x \)-direction.

a) Express \( \sin \theta(x) \) in terms of \( f'(x) \).

b) Show that \( \frac{d}{dx} \sin \theta(x) = \kappa_f(x) \).

c) Use the FTC to show that the total curvature of a rope depends only on the angle of the rope at the endpoints; if you keep these angles fixed and wiggle the rope in the middle, the total curvature won’t change.