Recall that the factorial $s!$ of any real number $s > -1$ is defined as the integral

$$s! = \int_{0}^{\infty} x^s e^{-x} \, dx,$$

and that $s! = s \cdot (s - 1)!$. If $s$ is a positive integer then $s! = s(s - 1) \cdot 2 \cdot 1$. And as we showed in class, $(1/2)! = \sqrt{\pi}/2$.

**Problem 1.** Compute $(5/2)!$, $(7/2)!$, and $(9/2)!$.

**Problem 2.** Compute $(-1/2)!$ in two ways: i) directly from the integral, and ii) from the recursion formula $s! = s \cdot (s - 1)!$.

**Problem 3.** Show that for any integer $k \geq 0$ we have

$$\left( k - \frac{1}{2} \right)! = k! P_k \sqrt{\pi},$$

where $P_k$ is the probability of getting $k$ heads when tossing a coin $2k$ times.

**Problem 4.** Express the Gaussian integral $G_n = \int_{0}^{\infty} x^n e^{-x^2} \, dx$ in terms of factorials. [Hint: do a $u$-substitution to turn it into a factorial integral.]

**Problem 5.** Let $v_n(r)$ be the volume of the sphere in $n$-dimensions. So $v_1(r)$ is the length of the interval $[-r, r]$, $v_2(r)$ is the area of a circle of radius $r$, $v_3(r)$ is the volume of the interior of the sphere of radius $r$, and $v_4(r)$ is the volume of the interior of the hypersphere in four dimensions (whose volume we computed in class, using Cavalieri’s principle).

Show that these lengths, areas, volumes, hypervolumes can be expressed in the single formula

$$v_n(r) = \frac{\pi^{n/2}}{(n/2)!} \cdot r^n, \quad n = 1, 2, 3, 4.$$

[In fact, this formula holds for any dimension $n \geq 0$.]

**Problem 6.** As Wallis guessed using his heuristic principle of interpolation, it is indeed true that

$$\int_{0}^{1} (1 - x^{1/p})^q \, dx = \frac{p! \cdot q!}{(p + q)!}$$

for all real numbers $p, q > -1$. You proved this on hw 7 for $p, q$ positive integers. In class, we proved this for $p = q = 1/2$. Show that this formula is also true for $p = 1/2$, $q = -1/2$, by computing both sides separately.
Problem 7. Turn each of the following integrals into a factorial integral and compute it in terms of factorials.

a) \( \int_0^\infty e^{-ax^2} \, dx \) (where \( a \) is a positive constant).

b) \( \int_0^\infty e^{-x^3} \, dx \)

c) \( \int_0^1 \frac{dx}{\sqrt{-\log x}} \)

The remaining exercises are about the error function \( \text{erf}(x) \), which is defined by

\[
\text{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-x^2/2} \, dx.
\]

(See Wallis-Gauss notes, section 6.)

Problem 8. Show that \( \text{erf}(x) \) has the following properties.

a) \( \text{erf}(0) = 0 \).

b) \( \lim_{x \to \infty} \text{erf}(x) = \frac{1}{2} \).

c) \( \text{erf}(x) \) is an odd function. That is, \( \text{erf}(-x) = -\text{erf}(x) \).

d) \( \text{erf}(x) \) is always increasing, is concave up for \( x < 0 \), and concave down for \( x > 0 \).

Problem 9. Show that

\[
\frac{1}{\sigma\sqrt{2\pi}} \int_a^b \exp \left( -\frac{1}{2} \left[ \frac{k - \mu}{\sigma} \right]^2 \right) \, dx = \text{erf} \left( \frac{b - \mu}{\sigma} \right) - \text{erf} \left( \frac{a - \mu}{\sigma} \right).
\]

Problem 10. For \( n = 100 \) tosses, find the approximate probabilities of getting

a) between 50 and 60 heads
b) at least 45 heads

Problem 11. For \( n = 10,000 \) tosses, find the approximate probabilities of getting

a) between 4900 and 5100 heads
b) no more than 4500 heads

Problem 12. Please do Exercise 5.6 in the Wallis-Gauss notes.