Problem 1. Determine whether the following series converge or diverge, and find the sum of those that converge.

a) \[\frac{1}{10000} + \frac{1}{10001} + \frac{1}{10002} + \frac{1}{10003} + \cdots\]

b) \[\frac{1}{10000} + \frac{1}{20000} + \frac{1}{30000} + \frac{1}{40000} + \cdots\]

c) \[\frac{1}{10000} + \frac{1}{20000} + \frac{1}{40000} + \frac{1}{80000} + \cdots\]

Problem 2. Show that the following series converge and find their sums.

a) \[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \cdots\]

Hint: Consider the series \((1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots)\) and \((1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots)\).

b) \[\frac{1}{4} + \frac{1}{10} + \frac{1}{18} + \frac{1}{28} + \frac{1}{40} + \frac{1}{54} + \frac{1}{70} + \cdots\]

Problem 3. The \(n^{th}\) Harmonic number \(H_n\) is defined to be the \(n^{th}\) partial sum of the harmonic series. That is, \(H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}\).

a) Show that \(\log(1 + n) < H_n < 1 + \log(n)\).

b) Use part a) to show that \(H_{1000000} < 15\).

c) Use part a) to show that
\[\lim_{n \to \infty} \frac{H_n}{\log n} = 1.\]
Problem 4. The goal of this exercise is to sum the series $x + 2x^2 + 3x^3 + \cdots$.

a) Differentiate both sides of the identity

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x},$$

being sure to simplify the right-hand side.

b) Use your result in a) to find a similar identity for

$$x + 2x^2 + 3x^3 + \cdots + nx^n$$

c) Use your result in b) to show that the series

$$\sum_{k=1}^{\infty} kx^k$$

converges if $|x| < 1$ and find the sum (a function of $x$) in that case.

d) Use your formula in c) to find the sum (now a number)

$$\sum_{k=1}^{\infty} \frac{k}{2^k}.$$  

Problem 5. Pretend you are Isaac Newton and you’ve just discovered the series

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

(as in Wallis-Gauss notes section 8). Now use the same method to find a similar series for arctan $x$. [Since you are Newton, you show no concern for convergence! But you do remember to intone “it may appear” at the appropriate times.]