MT310 Homework 1

Solutions

January 29, 2010

Exercise 1. Let \( G = \{e, x, y\} \) be any group with three elements. Without knowing the group law, fill in the Cayley table.

Solution. The product \( xy \) must be one of \( e, x, y \). If \( xy = x \) then \( y = e \), which it is not. Likewise, \( xy \neq y \). Hence \( xy = e \) and \( y = x^{-1} \). Now \( x^2 \neq x \) (lest \( x = e \)) and \( x^2 \neq e \), (lest \( x = x^{-1} = y \)) so \( x^2 = y \). Likewise \( yg = e \) and \( y^2 = x \). Hence \( G \) has multiplication table

\[
\begin{array}{c|ccc}
\circ & e & x & y \\
\hline
e & e & x & y \\
x & x & y & e \\
y & y & e & x \\
\end{array}
\]

Exercise 2. Let \( G = \{e, x, y, z\} \) be a group with four elements. Again, you are not told the group law. Show that there are exactly two possibilities for the Cayley table.

Solution. If \( x^2 = y^2 = z^2 = e \) then the product of any two of these is the third, and we get the table on the left. Otherwise, some element, say \( x \), does not square to \( e \). Then \( x^2 \) is either \( y \) or \( z \), say \( y \). Now \( xy \) is either \( e \) or \( z \). But if \( xy = e \) then \( xz \neq e \) (lest \( z = x^{-1} = y \)) and \( xz \neq y \) (lest \( z = x \)), and \( xz \neq x, z \) as before. So we cannot have \( xy = e \), so \( xy = z \). We now have \( x^2 = y \), \( x^3 = z \), so \( x^4 = e \). We see that \( G \) is cyclic, generated by \( x \), and get the table on the right. If you take a different element to be one not squaring to \( e \), then you get the same table, but with the rows and columns permuted.

\[
\begin{array}{c|cccc}
\circ & e & x & y & z \\
\hline
e & e & x & y & z \\
x & x & e & z & y \\
y & y & z & e & x \\
z & z & y & e & x \\
\end{array}
\]

Exercise 3. Let \( G \) be a group and let \( g_1, g_2, \ldots, g_n \) be elements of \( G \). Prove that

\[(g_1g_2\cdots g_n)^{-1} = g_n^{-1}g_{n-1}^{-1}\cdots g_2^{-1}g_1^{-1}.\]

Proof. It is obvious for \( n = 1 \). For \( n = 2 \), we have

\[(g_2^{-1} \cdot g_1^{-1})(g_1 \cdot g_2) = g_2^{-1} \cdot e \cdot g_2 = g_2^{-1}g_2 = e,
\]
and likewise $(g_1 \cdot g_2)(g_2^{-1} \cdot g_1^{-1}) = e$. Suppose now that $n \geq 2$. Let $g = g_1 g_2 \cdots g_{n-1}$. By induction, we have

$$g^{-1} = g_{n-1}^{-1} \cdots g_1^{-1}.$$  

From the case $n = 2$, we have

$$(g_1 g_2 \cdots g_n)^{-1} = (g \cdot g_n)^{-1} = g_n^{-1} g^{-1} = g_n^{-1} g_{n-1}^{-1} \cdots g_1^{-1}.$$  

Exercise 4. Let $\mathbb{Z}_n^\times$ be the group of units of $\mathbb{Z}_n$ and assume that $n \geq 3$. Prove that there is an element $a \in \mathbb{Z}_n^\times$ such that $a^2 = 1$, but $a \neq 1$.

Proof. Taking $a = [-1]$ does the job: we have $[-1]^2 = [(-1)^2] = [1]$, and $-1 \neq 1 \mod n$ since $n \geq 3$.

Exercise 5. Let $G$ be a group for which $g^2 = e$ for all $g \in G$. Prove that $G$ is abelian.

Proof. Let $x, y \in G$. We have $x^2 = y^2 = (xy)^2 = e$. Multiplying both sides of the equation

$$e = (xy)(xy)$$

on the left by $x$ and on the right by $y$ gives

$$xy = x(xy)(xy)y = (x^2)(yx)(y^2) = e(yx)e = yx.$$  

Hence $xy = yx$ for all $x, y \in G$ so $G$ is abelian.

Exercise 6. Let $G$ be the symmetry group of an equilateral triangle, and let $a, b \in G$ be two reflections. Write the remaining three non-identity elements of $G$ in terms of $a$ and $b$.

Solution. We have $G = \{e, a, b, aba, ab, ba\}$. The third reflection is $aba = bab$ and the two rotations of order three are $ab, ba$.