Exercise 1. Suppose \(G\) has two subgroups \(H, K\) with \(K \triangleleft G\). Let \(HK = \{hk : h \in H, k \in K\}\). Prove that \(HK\) is a subgroup of \(G\).

Proof. Let \(h_1k_1\) and \(h_2k_2\) be two elements of \(HK\). Then

\[
(h_1k_1)(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1} = h_1h_2^{-1}k_1k_2^{-1}.
\]

We have \(h_1h_2^{-1} \in H\) and since \(K \triangleleft G\) we have \(k_1k_2^{-1}h_2^{-1} \in K\). Hence \(h_1k_1(h_2k_2)^{-1} \in HK\), so \(HK\) is a subgroup of \(G\).

\(\square\)

Exercise 2. For each of the groups \(G = S_3, A_4, S_4\), find subgroups \(H, K\) with \(K \triangleleft G\), such that \(G = HK\). Prove your claims.

Solution.

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(There are two possible answers for \(S_4\).) \(\square\)

Exercise 3. This exercise shows how to recognize direct products. Let \(G\) be a group with normal subgroups \(H \triangleleft G\) and \(K \triangleleft G\). Assume that \(H \cap K = \{e\}\) and \(HK = G\). Prove that \(G \cong H \times K\).

Hint: First show that \(hkh^{-1}k^{-1} = e\) for all \(h \in H, k \in K\). Then show that the function \(f : H \times K \to G\) given by \(f(h, k) = hk\) is a group isomorphism.

Proof. For all \(h \in H, k \in K\), we have

\[
hkh^{-1}k^{-1} = (hkh^{-1})k^{-1} = h(kh^{-1}k^{-1}) \in H \cap K,
\]

so \(hkh^{-1}k^{-1} = e\). This means that \(hk = kh\).

Now prove that \(f\) is an isomorphism, as follows. For \(h, k' \in H\) and \(k, k' \in K\), we compute:

\[
f(h, k) \cdot f(h', k') = f((hk)(h'k')) = h(kh')(k') = h(h'k')(k') = f(hh', kk') = f((h, k) \cdot (h', k')).
\]

Hence \(f\) is a homomorphism. Since \(HK = G\), any element \(g \in G\) may be written as \(g = hk\) for some \(h \in H, k \in K\), and \(f(h, k) = hk = g\), so \(f\) is surjective. Finally, if \((h, k) \in \ker f\), then \(hk = e\), so \(h = k^{-1} \in H \cap K = \{e\}\), so \((h, k) = (e, e)\). Hence \(f\) is injective. \(\square\)

Exercise 4. Let \(G\) be a nonabelian group of order \(2n\), where \(n \geq 3\). Suppose there exist elements \(a, b \in G\) such that \(a\) has order \(n\), \(b\) has order \(2\), and \(bab^{-1} = a^{-1}\). Prove that \(G \cong D_n\).

Hint: First show that \(G = \{a^ib^j : 0 \leq i < n, 0 \leq j \leq 1\}\), then find elements in \(D_n\) analogous to \(a, b\) and use all this to define an isomorphism \(f : G \to D_n\).

Proof. The subgroup \(\langle a \rangle\) has index two in \(G\). It does not contain \(b\), lest \(a = bab^{-1} = a^{-1}\), contradicting \(n \geq 3\). Hence

\[
G = \langle a \rangle \cup \langle a \rangle b = \{a^ib^j : 0 \leq i < n, 0 \leq j \leq 1\}.
\]
In $D_n$, let $r$ be a rotation by $2\pi/n$ and let $s$ be a reflection. Then $r$ has order $n$ and $s$ has order 2. Using a picture, you can check that $ss^{-1} = r^{-1}$. Hence we also have

$$D_n = \{ r^is^j : 0 \leq i < n, 0 \leq j \leq 1 \}.$$ 

Define $f : G \to D_n$ by $f(a^ib^i) = r^is^j$. We must show that it is a group homomorphism. Now

$$a^ib^ia^k/b^j = \begin{cases} a^{i+k}b^j & \text{if } j = 0 \\ a^{i-k}b^{j+1} & \text{if } j = 1. \end{cases}$$

So

$$f(a^ib^ia^k/b^j) = \begin{cases} r^{i+k}s^j & \text{if } j = 0 \\ r^{i-k}s^{j+1} & \text{if } j = 1. \end{cases}$$

$$= r^is^j r^k s^j$$

$$= f(a^ib^i)f(a^k/b^j).$$

Therefore $f$ is a group homomorphism, as claimed. It is clear that $f$ is a bijection, so it is an isomorphism. \hfill \Box

**Exercise 5.** Suppose that $G$ is a nonabelian group of order $2p$, where $p > 2$ is a prime. 

Prove that $G \simeq D_p$.

Hint: Find elements $a, b$ as in the previous exercise.

**Proof.** Since $G$ has even order, it contains an element $b$ of order 2. (Proved in exam 1 study problems.) Since $G$ is nonabelian, it cannot have all elements of order 2 (proved in hw), nor can it have an element of order $2p$, lest it be cyclic. Hence $G$ has an element $a$ of order $p$. The subgroup $\langle a \rangle$ has index two in $G$, hence is normal, so $bab^{-1} = a^k$ for some $k$. Conjugating again by $b$, we have

$$a = b^2ab^{-2} = (a^k)^{k} = a^{2k}.$$ 

hence $p \mid k^2 - 1$. Since $p$ is prime, we have $p \mid k - 1$ or $p \mid k + 1$. But if $p \mid k - 1$ then $bab^{-1} = a^{k} = a$ so $G$ would be abelian, which it is not. Hence $p \mid k + 1$, so $bab^{-1} = a^{-1}$. By the result in the previous problem, we have $G \simeq D_{2p}$. \hfill \Box

**Exercise 6.** Let $G$ be a finite group, let $a \in G$, and let $A$ be the conjugacy class of $a$ in $G$, and let $H = C_G(a)$ be the centralizer of $a$ in $G$. Prove that

$$|A| = \frac{|G|}{|H|}.$$ 

Hint: Show that the function $f : G/H \to A$ given by $f(gH) = gag^{-1}$ is a well-defined bijection.

**Proof.** We first show that $f$ is well-defined. Suppose $gH = kH$. Then $g = kh$ for some $h \in H$. We have $f(gH) = gag^{-1} = khah^{-1}k^{-1}$. Since $h \in H = C_G(a)$, we have $hah^{-1} = a$. Therefore $f(gH) = kak^{-1} = f(kH)$, so $f$ is well-defined. By definition, $A = \{ gag^{-1} : g \in G \}$. So if $b \in A$ there is $g \in G$ such that $b = gag^{-1}$, and $f(gH) = gag^{-1} = b$. Hence $f$ is surjective. Finally, suppose $f(gH) = f(kH)$, for some $g, k \in G$. Then $gag^{-1} = kak^{-1}$, so $k^{-1}gag^{-1}k^{-1} = a$. This means $k^{-1}g \in H$, so $kH = gH$. Hence $f$ is injective. Since $f : G/H \to A$ is a bijection, we have $|A| = |G/H| = |G|/|H|$. \hfill \Box

Comment: Note that $H$ is not necessarily normal in $G$. The set $G/H$ is not a group, and $f$ is not a group homomorphism. That’s why we could not prove injectivity by showing ker $f$ is trivial.

**Exercise 7.** Use the first isomorphism theorem to prove that $S_4/K_4 \simeq S_3$.

**Proof.** It suffices to find a surjective homomorphism $f : S_4 \to S_3$ with ker $f = K_4$. We have found such a homomorphism in Exercise 5 of hw 3 (see also Exercise 6 of hw 3). By the first isomorphism theorem, it follows that the map

$$\bar{f} : S_4/K_4 \to S_3$$

given by $\bar{f}(\sigma K_4) = f(\sigma)$, for $\sigma \in S_4$, is an isomorphism. \hfill \Box