MT310 Exam 1 study guide/ problems

Definitions to know:
• group
• subgroup
• order of a group and element of a group
• homomorphism
• isomorphism, homomorphism, kernel, image
• conjugacy class, centralizer
• left and right coset, index of a subgroup.

Groups to know in detail:  (Order of the group and its elements, all of the subgroups, conjugacy classes, homomorphisms to other groups, various incarnations.)
• $\mathbb{Z}_n$, cyclic groups
• $\mathbb{Z}_2 \times \mathbb{Z}_2$
• $\mathbb{Z}_n^\times$, also known as $U(n)$ in the text, and $(\mathbb{Z}/n\mathbb{Z})^\times$ elsewhere.
• $S_3$ and $S_4$
• $D_4$
• $A_4$.

Permutation Groups:  Know how to:
• Calculate products and conjugacy classes in $S_n$
• Construct and analyze homomorphisms from $f: G \to S_n$, arising from permutation actions of a group $G$ on a set with $n$-elements.

Practice problems:
1. Prove that any group of order $\leq 5$ is abelian.
2. Let $H, K$ be subgroups of a group $G$. Prove that $H \cap K$ is a subgroup of $G$. Give an example of a specific $G, H, K$ for which $H \cup K$ is not a subgroup of $G$.
3. Prove that $D_3$, the symmetry group of an equilateral triangle, is isomorphic to the symmetric group $S_3$.
4. Find two groups of order 6 which are not isomorphic to each other and prove that they are not isomorphic to each other.
5. Prove that if $|G| = p$ is prime then $G$ is isomorphic to $\mathbb{Z}_p$.
6. The center of a group $G$ is defined as $Z(G) = \{a \in G: \ ab = ba \ \forall b \in G\}$. Prove that $Z(G)$ is a subgroup of $G$. Find $Z(G)$ for all the groups you know.
7. List as many non-isomorphic groups of order 8 as you can, and give conditions for recognizing each of them.

8. Prove that if $|G|$ is even then $G$ has an element of order 2.

9. Prove that if $a, b \in G$ commute and have relatively prime orders $m, n$, then the order of $ab$ is $mn$.

10. Suppose $a \in G$ is an element of odd order $m$. Prove that $a^2$ also has order $m$.

11. Suppose $\sigma \in S_n$ is a cycle of odd length $m$. Prove that $\sigma^2$ is also a cycle of length $m$.

12. If $\sigma \in S_n$ is a cycle of even length $2k$, what is the cycle type of $\sigma^2$?

13. Define $f : \mathbb{Z}_9^\times \to \mathbb{Z}_9^\times$ by $f(a) = a^3$, for any $a \in \mathbb{Z}_9^\times$. Prove that $f$ is a well-defined homomorphism and compute $\ker f$ and $\text{im} f$.

14. A subgroup $H < S_n$ is called transitive if for all $i, j \in \{1, 2, \ldots, n\}$, there exists $h \in H$ such that $h(i) = j$. Find a transitive subgroup $H < S_6$ which is isomorphic to $S_4$ and list the cycle types in $H$. (Hint: $S_4$ acts on the cube.)