Exercise 1. If group $G$ acts on a set $X$ and $H \leq G$, let $X^H = \{ x \in X : h \cdot x = x \}$ denote the fixed-point set of $H$ in $X$.

a) Prove that the normalizer $N_G(H)$ of $H$ in $G$ preserves $X^H$. [That is, prove that $n \cdot x \in X^H$ for all $n \in N_G(H)$ and $x \in X^H$.]

b) Prove that the center of $S_X$ is trivial if $|X| \geq 3$. [Hint: Let $x \in X$ and apply a) to the group $G = S_X$, with $H = S_{X - \{x\}}$ the subgroup of $G$ fixing $x$. Note that we do not require $X$ to be finite.]

Exercise 2. Let $F$ be a field, let $G = GL_n(F)$, let $T$ be the subgroup of diagonal matrices in $G$, and let $N$ be the normalizer of $T$ in $G$. Prove that $N/T \simeq S_n$. [Hint: Consider the fixed-points of $T$ in $\mathbb{P}^{n-1}(F)$.]

Exercise 3. Prove that each of the following groups is isomorphic to $S_4$, the symmetric group on four letters. Do not use the classification of groups of order 24. Rather, find in each case an isomorphism $f : G \to S_4$ arising from an action of $G$ on a set with four elements.

a) $G = PGL_2(3)$.
b) $G$ is the group of orientation-preserving symmetries of the cube.
c) $G$ is the group of all symmetries of the tetrahedron.
d) $G$ is the automorphism group of $S_4$.
e) $G$ is a group of order 24 in which no Sylow subgroup is normal.

Proof.

Exercise 4. Suppose $H$ is a subgroup of the symmetric group $S_n$ with $[S_n : H] = n$. Prove the following.

a) $H \simeq S_{n-1}$. [Consider the action of $S_n$ on $S_n/H$.]
b) If $S_n$ has a transitive subgroup of index $n$ then $S_n$ has an automorphism which is not inner.
c) $S_5$ has 6 subgroups of order 5.
d) $S_6$ has an automorphism which is not inner.

Exercise 5. Compute the orders of $PSL_3(4)$ and $PSL_4(2)$. Then show that these groups are not isomorphic. [Hint: Consider the normalizer of a Sylow 2-subgroup.]

Exercise 6. Determine the structure of a Sylow 2-subgroup of $SL_2(3)$ and use this to decide whether $SL_2(3) \simeq S_4$ or not.
Exercise 7. Complete the following alternative proof of Cauchy’s theorem stating that if $p$ divides the order of a group $G$ then $G$ has an element of order $p$:
Let $X = \{(g_0, g_1, \ldots, g_{p-1}) : g_0g_1\cdots g_{p-1} = 1\}$ and let $\mathbb{Z}/p\mathbb{Z}$ act on $X$ by $k \cdot (g_0, g_1, \ldots, g_{p-1}) = (g_k, g_{k+1}, \ldots, g_{k+p-1})$, where the subscripts are read modulo $p$. Consider fixed points.

Exercise 8. Let $G$ be a group of order $m \cdot p^r$, where $m \leq 5$, $p$ is a prime not dividing $m$, and $r$ is any positive integer. Prove that $G$ is not simple.

Exercise 9. Let $p$ be the smallest prime dividing $|G|$. If the Sylow $p$-subgroups in $G$ are isomorphic to $C_p \times C_p$ then either $G$ has a normal $p$-complement or the following holds: $p = 2$ and $|N_G(P)/C_G(P)| = 3$ and $G$ has a unique conjugacy class of elements of order two.

Exercise 10. Prove that no group of order $< 60$ is simple. You are allowed to consider two particular orders, and the rest of the orders must be ruled out by general results (such as the previous two exercises).