Proposition IV.5

How to circumscribe a circle about a given triangle.

According to Definition IV.6, a circle is circumscribed about a figure if the circumference of the circle passes through each angle of the figure.

Let $\triangle ABC$ be the given triangle. Bisect $AB$ at point $D$ and $AC$ at $E$ (Prop. I.10). Draw the perpendiculars to $AB$ at $D$ and to $AC$ at $E$ (Prop. I.11), meeting at point $F$. We distinguish three cases, based on the position of $F$. 
Case 1: $F$ is inside $\triangle ABC$.

Draw $AF$, $BF$, and $CF$. Then $\triangle ADF \cong \triangle BDF$ (Prop. 1.4, SAS), so $AF = BF$. Similarly, $\triangle AEF \cong \triangle CEF$, so $AF = CF$. We conclude that $AF = BF = CF$, and then draw the circle of that radius centered at $F$.

Case 2: $F$ is on $BC$.

Draw $AF$. A similar argument shows that $AF = BF = CF$, so we again draw the circle of that radius centered at $F$. 
**Case 3:** $F$ is outside $\triangle ABC$.

Draw $AF$, $BF$, and $CF$. Again, $\triangle ADF \cong \triangle BDF$, so $BF = AF$, and $\triangle AEF \cong \triangle CEF$, so $AF = CF$. Draw the circle of radius $AF = BF = CF$ centered at $F$.

Q.E.F.

**Comments:**

1. Only Propositions I.4, I.10, and I.11 are used, so this could have been presented much earlier.

2. Euclid comments that in Case 1, $\angle BAC$ is acute, in Case 2 right, and in Case 3 obtuse.
3. Is the circle unique? Yes. We could appeal to Proposition III.10: *A circle does not cut a circle at more points than two.* Or we could argue as follows:

Draw a circumscribing circle with center at $F$.

Since $E$ bisects $AC$, we have that $FE \perp AC$ by Prop. III.3. But similarly, $FD \perp AB$. Thus $F$, as the intersection of the perpendicular bisectors of two sides of the triangle, is unique.

4. How many circles pass through a given number of points?

Through one point, we can draw a circle $C(d, \theta)$ of any diameter $d$ at any angle $\theta$: *two dimensions* of choice.
We can visualize these choices as points on a cylinder (a 2-dimensional geometric object):

Through two points, we can draw circles $C(d)$ of any diameter $d$: one dimension of choice.

The choice corresponds to a point on the perpendicular bisector of the segment between the two points (a 1-dimensional geometric object).

By this Proposition, through three noncollinear points, determining a triangle, there is one (circumscribed) circle, corresponding to a point, or $0$ dimensions of choice.

Through four (or more) points, there need not be a circle. To build such a set of points, given three noncollinear points, construct the circle through them, and then the choose a fourth point not on the circle.