Chapter 16. Solutions to Exercises

Exercise 16.1 Find the eigenvalues and eigenvectors of the following matrices.

(a) \( A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 4 \end{bmatrix} \)

(b) \( A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \)

(c) \( A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{bmatrix} \)

(d) \( A = \begin{bmatrix} 1/2 & -1/6 & 1/6 \\ 1/2 & -1/2 & -1/2 \\ -2/5 & 1/5 & 0 \end{bmatrix} \)

(e) \( A = \begin{bmatrix} -3 & -3 & 15 \\ -14 & 16 & 10 \\ -7 & -1 & 23 \end{bmatrix} \)

Solutions: Unless otherwise specified, we give the eigenvectors and eigenvalues in the same order. When the eigenspace is a line we give a nonzero vector on the line; your answers may differ from those below by a scalar.

(a) Eigenvalues: 1, 2, 3. Eigenvectors: \((2,1,1), (1,1,1), (1,1,2)\).

(b) Eigenvalues: 0, 0, 0. Eigenvectors: \(E(0)\) is the plane \(y+z = 0\).

(c) Eigenvalues: 1, 0, −1. Eigenvectors: \((2,1,1), (1,1,1), (1,1,2)\).

(d) Eigenvalues: 0, 0, 0. Eigenvector: \((-1,-2,1)\).

(e) Eigenvalues: 18, 18, 0. Eigenvectors: \(E(18)\) is the plane \(7x + y - 5z = 0\), 
\(E(0)\) is line thru \((3,2,1)\).

Exercise 16.2 Let \( A \) be a 3 × 3 matrix with eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \). Express the coefficients of the characteristic polynomial \( P_A(x) \) in terms of \( \lambda_1, \lambda_2, \lambda_3 \).

Solution: The coefficients of \( P_A(x) \) are the various sums of diagonal determinants:

\[ P_A(x) = x^3 - \text{tr}(A)x^2 + (\det A_{11} + \det A_{22} + \det A_{33})x - \det(A). \]  (1)

But also, since the roots of \( P_A(x) \) are \( \lambda_1, \lambda_2, \lambda_3 \), we have

\[ P_A(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3). \]

Multiplying this out, we get

\[ P_A(x) = x^3 - (\lambda_1 + \lambda_2 + \lambda_3)x^2 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)x - \lambda_1\lambda_2\lambda_3. \]  (2)
Comparing the coefficients in (1) and (2), we find that

\[
\begin{align*}
\text{tr}(A) &= \lambda_1 + \lambda_2 + \lambda_3 \\
\det A_{11} + \det A_{22} + \det A_{33} &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\
\det(A) &= \lambda_1\lambda_2\lambda_3.
\end{align*}
\]

**Exercise 16.3** A 3 × 3 migration matrix \( A \) has all entries between 0 and 1, and the sum of each column is 1. Thus \( A \) looks like

\[
A = \begin{bmatrix}
1 - x - y & u & v \\
x & 1 - u - z & w \\
y & z & 1 - v - w
\end{bmatrix}.
\]

Let \( P \) denote the initial population. Find the stable population distribution.

Hint: The stable population vector is fixed by \( A \). Your answer will be a vector involving \( x, y, z, w, u, v \).

**Solutions:**

We need a vector on the line

\[
\ker [A - I] = \ker \begin{bmatrix}
-x - y & u & v \\
x & -u - z & w \\
y & z & -v - w
\end{bmatrix}
\]

whose sum of coordinates is \( P \). Take the cross-product of two rows, divide by the coordinate sum and multiply by \( P \). Using the first two rows, we get

\[
P(uw + zv + uv, xv + xw + yw, xz + yu + yz) \\
(uw + zv + uv + x + xw + yw + xz + yu + yz)
\]