A vector space \( V \) is called \textit{finite dimensional} if \( V \) is spanned by a finite set of vectors. So \( \mathbb{R}^n \) and \( P_n \) are finite dimensional, while the vector space \( P \) of all polynomials is infinite dimensional. \(^1\) In this section we will assume that \( V \) is a finite dimensional vector space and that \( V \) is nontrivial (that is, \( V \) contains nonzero vectors).

To define the dimension of \( V \), we need the following fundamental fact.

**The Basis Theorem:** Every nontrivial finite dimensional vector space \( V \) has a basis and all bases of \( V \) contain the same number of vectors.

The Basis Theorem allows us to define dimension precisely:

**Definition of dimension:** The \textbf{dimension} of a finite dimensional vector space \( V \), denoted \( \dim V \), is the number of vectors in any basis of \( V \).

To compute the dimension of a vector space it is enough to find one basis. The Basis Theorem guarantees that all other bases will have the same number of elements as the one you found.

**Examples:**

- Clearly \( \mathbb{R}^n \) has dimension \( n \), because \( \{e_1, \ldots, e_n\} \) is a basis.
- The hyperplane \( V \) defined by \( x + y + z + w = 0 \) in \( \mathbb{R}^4 \) has \( \dim V = 3 \), since it has the basis \( \{e_1 - e_2, e_2 - e_3, e_3 - e_4\} \).
- The vector space \( W \) defined by the same equation \( x + y + z + w = 0 \) in \( \mathbb{R}^5 \) has \( \dim W = 4 \), since it has the basis \( \{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_5\} \).
- The vector space \( P_n \) of polynomials of degree \( \leq n \) has dimension \( n + 1 \). We know of two bases:
  \[ \{1, x, \ldots, x^n\}, \quad \text{and} \quad \{P_0, P_1, \ldots, P_n\}. \]
  According to the Basis Theorem, all bases of \( P_n \) have \( n + 1 \) elements.
- Let \( V \) be the vector space of solutions to the differential equation
  \[ f'' + af' + bf = 0, \]
  where \( a \) and \( b \) are constants such that the roots \( \lambda \) and \( \mu \) of the polynomial \( x^2 + ax + b \) are distinct. In the previous section we proved that \( \{e^{\lambda x}, e^{\mu x}\} \) is a basis of \( V \). According to the Basis Theorem, all bases of \( V \) have two elements and \( \dim V = 2 \). For example, the solution space of \( f'' + f = 0 \) has bases
  \[ \{e^{ix}, e^{-ix}\} \quad \text{and} \quad \{\cos x, \sin x\}. \]

The aim of this section is to prove the Basis Theorem. We fix a nontrivial finite dimensional vector space \( V \), and vectors \( v_i, w_i \) will always denote nonzero vectors in \( V \).

\(^1\)Note that we have defined “finite dimensional” without yet defining “dimension”.


Lemma 1: If \( \{v_1, \ldots, v_m\} \) is linearly dependent, then there exists \( j \leq m \) such that \( v_j \) is a linear combination of \( \{v_1, \ldots, v_{j-1}\} \).

Proof: Since \( \{v_1, \ldots, v_m\} \) is linearly dependent, there are scalars, \( c_1, \ldots, c_m \), not all zero, such that \( c_1 v_1 + \cdots + c_m v_m = 0 \). Let \( j \) be the largest index such that \( c_j \neq 0 \). Then \( c_1 v_1 + \cdots + c_j v_j = 0 \), and

\[
v_j = -\frac{c_1}{c_j} v_1 + \cdots + \frac{c_{j-1}}{c_j} v_{j-1},
\]

so \( v_j \) is a linear combination of \( \{v_1, \ldots, v_{j-1}\} \). ■

Lemma 2: If \( \{v_1, \ldots, v_m\} \) spans \( V \) and for some \( k \leq m \) the first \( k \) of these vectors, \( v_1, \ldots, v_k \), are linearly independent, then by possibly discarding some vectors \( v_i \) for \( i > k \) we can obtain a basis of \( V \) which contains \( \{v_1, \ldots, v_k\} \).

Proof: If \( \{v_1, \ldots, v_m\} \) is linearly independent then it is a basis of \( V \) already and we don’t need to discard any vectors. So assume \( \{v_1, \ldots, v_m\} \) is linearly dependent. By Lemma 1, when we sweep through the list

\[
v_1, v_2, \ldots, v_m
\]

from left to right, we will find a vector which is a linear combination of the previous ones. Call such a vector “bad”. Let \( v_a \) be the first bad vector encountered and discard it. Note that

- \( a > k \) since \( \{v_1, \ldots, v_k\} \) is assumed to be linearly independent.
- \( \{v_1, \ldots, v_k, \ldots, v_{a-1}\} \) is linearly independent, since \( v_a \) was the first bad vector encountered.
- \( \{v_1, \ldots, v_k, \ldots, v_{a-1}, v_{a+1}, \ldots, v_m\} \) spans \( V \), since \( v_a \) is in the span of this set, so we can express every element of \( V \) without using \( v_a \).

If the new set (with the bad vector \( v_a \) removed) is linearly independent, we have a basis. Otherwise repeat the sweep through the list with \( v_a \) removed and again discard the first bad vector, call it \( v_b \), that is a linear combination of the previous ones. As we iterate this procedure, the bad vectors move progressively to the right on our list, so we will eventually run out of bad vectors. Thus we arrive a linearly independent set

\[
\{v_1, \ldots, v_k, \ldots, v_{a-1}, \ldots, v_{b-1}, \ldots\}
\]

that still spans \( V \), hence is a basis of \( V \) containing \( \{v_1, \ldots, v_k\} \), as desired. ■

As a consequence of Lemma 2 we have that

*Every spanning set may be shrunk down to a basis.*

More precisely:

**Proposition 1:** Every nontrivial finite dimensional vector space \( V \) has a basis, and if \( S \subset V \) spans \( V \) then some subset of \( S \) is a basis of \( V \).

Proof: Since \( V \) is finite dimensional it has a finite spanning set. If \( S \) is any spanning set of \( V \), it must contain a nonzero vector, since \( V \) is nontrivial. Let \( v_1 \) be a nonzero vector in \( S \). By Lemma 2 (with \( k = 1 \)), there is a basis of \( V \) contained in \( S \) and containing \( v_1 \). ■
We need one more lemma to finish the proof of the basis theorem.

**Lemma 3:** Suppose \( \{v_1, \ldots, v_n\} \) is a basis of \( V \) and \( \{w_1, \ldots, w_m\} \) is a linearly independent subset of \( V \). Then \( m \leq n \).

**Proof:** Since \( \{v_1, \ldots, v_n\} \) spans \( V \), the vector \( w_m \) can be written as a linear combination of the \( v_i \)'s, so the set \( \{w_m, v_1, \ldots, v_n\} \) is linearly dependent. Applying Lemma 2 (with \( k = 1 \)), we may discard some \( v_i \)'s to obtain a basis \( \{w_m, v_{i_1}, \ldots, v_{i_\ell}\} \), with \( \ell < n \). Since \( w_{m-1} \) is in the span of this new basis, the set \( \{w_{m-1}, w_m, v_{i_1}, \ldots, v_{i_\ell}\} \) is linearly dependent, so by Lemma 2 (now with \( k = 2 \), since \( \{w_{m-1}, w_m\} \) is linearly independent) we may discard some \( v_{i_j} \)'s to obtain a basis containing \( w_{m-1}, w_m \) and some \( v_i \)'s. Repeating this \( m \) times we obtain a basis containing \( w_1, \ldots, w_m \) and possibly some remaining \( v_i \)'s. At each stage we have added exactly one \( w_j \) and removed at least one \( v_i \). So there must have been at least \( m v_j \)'s to begin with. That is, \( m \leq n \). ■

We can now prove our main result.

**Proof of the Basis Theorem:** We have already proved (see Prop. 1) that \( V \) has at least one basis. Suppose we have two bases \( \{v_1, \ldots, v_n\} \) and \( \{w_1, \ldots, w_m\} \) of \( V \). Applying Lemma 3 in both directions we have \( m \leq n \) and \( n \leq m \), so \( n = m \) and the Basis Theorem is proved. ■

As a consequence of the Basis Theorem, we have an analogue of Prop. 1.

*Every linearly independent set may be expanded to a basis.*

More precisely:

**Proposition 2:** If \( S \subset V \) is linearly independent then there is a basis of \( V \) containing \( S \).

**Proof:** By the Basis Theorem, \( V \) has a basis \( \{v_1, \ldots, v_n\} \). Suppose \( S = \{w_1, \ldots, w_k\} \subset V \) is linearly independent. Then the set \( S' = \{w_1, \ldots, w_k, v_1, \ldots, v_n\} \) spans \( V \) (since the \( v_i \)'s do), and by Lemma 2 be may obtain a basis \( B \) by removing some of the \( v_i \)'s from \( S' \). Since we do not remove any \( w_j \)'s this basis \( B \) contains \( S \). ■

As a consequence we have the *Goldilocks Principle*:

**Proposition 3:** Let \( V \) be a vector space of dimension \( n \) and let \( S \) be a subset of \( V \).

1. If \( S \) contains more than \( n \) vectors then \( S \) cannot be linearly independent.
2. If \( S \) contains fewer than \( n \) vectors then \( S \) cannot span \( V \).
3. If \( S \) contains exactly \( n \) vectors then the following are equivalent:
   (a) \( S \) is linearly independent
   (b) \( S \) spans \( V \)
   (c) \( S \) is a basis of \( V \).

**Proof:** If \( S \) is linearly independent then it can be expanded to a basis, which has exactly \( n \) vectors, so \( S \) cannot have more than \( n \) vectors to begin with. Likewise if \( S \) spans \( V \) then it can be shrunk down to a basis so it has at least \( n \) vectors to begin with.
Finally, if $S$ has exactly $n$ vectors then it has the same number of vectors that a basis does, so $S$ cannot be expanded or shrunk to a basis. Therefore if $S$ is linearly independent it must already be a basis of $V$ and similarly if $S$ spans $V$ it must already be a basis of $V$. ■

The Goldilocks principle means that if you already know $\dim V$ then there are fewer steps to proving that some subset $S \subset V$ is a basis. If $S$ contains exactly $\dim V$ vectors then it is enough to check either that $S$ is linearly independent or that $S$ spans $V$. Remember this shortcut is only allowed if you already know the dimension of $V$.

**Exercise 21.1**  Find a basis and the dimension of the hyperplane in $\mathbb{R}^5$ with equation $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 0$.

**Exercise 21.2**  Find a basis of the hyperplane in $\mathbb{R}^n$ with equation $x_1 + x_2 + \cdots + x_n = 0$.

**Exercise 21.3**  Find bases of the kernel and image of the $n \times m$ matrix $A$ all of whose entries are 1.

**Exercise 21.4**  Find a basis of $\mathbb{R}^4$ containing the vectors $(1, 2, 3, 4)$ and $(4, 3, 2, 1)$.

**Exercise 21.5**  Let $V$ be the hyperplane in $\mathbb{R}^4$ given by $x - y + z - w = 0$ and let

$S = \{(0, 1, 2, 1), (1, 2, 1, 0), (1, 2, 4, 3), (2, 1, 3, 4)\} \subset V$.

(a) Show that $V$ is spanned by $S$.

(b) Find a basis of $V$ that is contained in $S$.

**Exercise 21.6**  Prove that any set of $n$ orthogonal nonzero vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$.

**Exercise 21.7**  Let $V$ be the vector space of solutions of the differential equation $f'' - f = 0$. Show that $\{\cosh x, \sinh x\}$ is a basis of $V$.

**Exercise 21.8**  Find a basis of and the dimension of the vector space $V$ consisting of polynomials of degree $\leq n$ vanishing at 1:

$V = \{f \in \mathbb{P}_n : f(1) = 0\}$

[Hint: consider powers of $x - 1$.]