Exercise 21.1  Find a basis and the dimension of the hyperplane in $\mathbb{R}^5$ with equation

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 0.$$ 

Solution: One basis is

$$\{(2, -1, 0, 0, 0), (0, 3, -2, 0, 0), (0, 0, 4, -3, 0), (0, 0, 0, 5, -4)\}.$$ 

Check spanning and linear independence as before. There are four vectors in this basis so the hyperplane has dimension four. By the basis theorem, a valid solution consists of four linearly independent vectors satisfying the equation of the hyperplane.

Exercise 21.2  Find a basis of the hyperplane in $\mathbb{R}^n$ with equation $x_1 + x_2 + \cdots + x_n = 0$.

Solutions: One basis is

$$e_i - e_{i+1}, \quad i = 1, 2, \ldots, n-1.$$ 

Check spanning and linear independence as before. There are $n-1$ vectors in this basis so the hyperplane has dimension $n-1$. By the basis theorem, a valid solution consists of $n-1$ linearly independent vectors whose coordinates sum to zero.

Exercise 21.3  Find bases of the kernel and image of the $n \times m$ matrix $A$ all of whose entries are 1.

Solution: The image is the line through $(1, 1, \ldots, 1)$ and the kernel is the hyperplane in the previous problem.

Exercise 21.4  Find a basis of $\mathbb{R}^4$ containing the vectors $(1, 2, 3, 4)$ and $(4, 3, 2, 1)$.

Solution: The given vectors are linearly independent, so can be expanded to a basis. Since $\mathbb{R}^4$ has dimension four, we must add two vectors. For example,

$$\{(1, 2, 3, 4), (4, 3, 2, 1), (1, 0, 0, 0), (0, 1, 0, 0)\}$$

is one such basis. To prove this, it suffices, by Goldilocks, to show that the four vectors are linearly independent. Any linearly independent set of four vectors in $\mathbb{R}^4$ containing the two given vectors is a valid answer.

Exercise 21.5  Let $V$ be the hyperplane in $\mathbb{R}^4$ given by $x - y + z - w = 0$ and let

$$S = \{(0, 1, 2, 1), (1, 2, 1, 0), (1, 2, 4, 3), (2, 1, 3, 4)\} \subset V.$$ 

a) Show that $V$ is spanned by $S$.

b) Find a basis of $V$ that is contained in $S$. 

**Solution:** Since there are more than three vectors, we try to shrink down to a basis, by throwing one vector out, say the last one. Let \((x, y, z, w) \in V\) We seek scalars \(c_1, c_2, c_3\) such that
\[
c_1(0, 1, 2, 1) + c_2(1, 2, 1, 0) + c_3(1, 2, 4, 3) = (x, y, z, w).
\]
This translates to the equations
\[
\begin{align*}
  c_2 + c_3 &= x \\
  c_1 + 2c_2 + 2c_3 &= y \\
  2c_1 + c_2 + 4c_3 &= z \\
  c_1 + 3c_3 &= w.
\end{align*}
\]
We find
\[
  c_1 = y - 2x, \quad c_2 = (y - 2z)/3, \quad c_3 = (w - y + 2x)/3.
\]
This proves that the first three vectors span \(V\), so a) is done. Since \(\dim V = 3\), this means the first three vectors are a basis of \(V\), as well, so b) is done.

**Exercise 21.6** Prove that any set of \(n\) orthogonal nonzero vectors in \(\mathbb{R}^n\) is a basis of \(\mathbb{R}^n\).

**Solution:** Let \(\{v_1, \ldots, v_n\}\) be a set of \(n\) orthogonal nonzero vectors in \(\mathbb{R}^n\). Since \(\dim \mathbb{R}^n = n\), it suffices to show this set is linearly independent. Suppose
\[
c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0.
\]
For each \(k = 1, 2, \ldots, n\) we have
\[
0 = \langle v_k, 0 \rangle = \langle v_k, c_1v_1 + c_2v_2 + \cdots + c_nv_n \rangle = c_k\langle v_k, v_k \rangle,
\]
by orthogonality. Since each \(v_k \neq 0\), we must have \(c_k = 0\) for each \(k\), so \(\{v_1, \ldots, v_n\}\) is linearly independent and is therefore a basis of \(\mathbb{R}^n\).

Alternatively (or additionally) you can show that every vector \(v \in \mathbb{R}^n\) can be written as
\[
v = \sum_{k=1}^{n} \frac{\langle v, v_k \rangle}{\langle v_k, v_k \rangle} v_k,
\]
in the same manner as Exercise 20.4 part b). This shows that \(\{v_1, \ldots, v_n\}\) spans \(\mathbb{R}^n\).

**Exercise 21.7** Let \(V\) be the vector space of solutions of the differential equation \(f'' - f = 0\). Show that \(\{\cosh x, \sinh x\}\) is a basis of \(V\).

**Solution:** First check that both functions satisfy the equation. We know \(\dim V = 2\), so it suffices to check linear independence. Suppose
\[
c_1 \cosh x + c_2 \sinh x = 0.
\]
Differentiating gives
\[
c_1 \sinh x + c_2 \cosh x = 0.
\]
Multiplying the first equation by \(\cosh x\), the second by \(\sinh x\) and subtracting, we get
\[
0 = c_1(\cosh^2 x - \sinh^2 x) = c_1,
\]
by the hyperbolic trig identity \(\cosh^2 x - \sinh^2 x = 1\). So \(c_1 = 0\), and then \(c_2 \sinh x = 0\), so \(c_2 = 0\). Hence \(\{\cosh x, \sinh x\}\) linearly independent and is therefore a basis of \(V\).
Exercise 21.8  Find a basis and the dimension of the vector space $V$ consisting of polynomials of degree $\leq n$ vanishing at 1:

$$V = \{ f \in \mathbb{P}_n : f(1) = 0 \}$$

[Hint: consider powers of $x - 1$.]

Solution: I claim that $\{ x - 1, (x - 1)^2, \ldots, (x - 1)^n \}$ is a basis of $V$. Since we don’t yet know the dimension of $V$, we must check both spanning and linear independence.

For spanning, let $f \in V$ and consider the Taylor expansion of $f$ at $x = 1$:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!} (x - 1)^k.$$  

Since $f(1) = 0$, the sum actually starts at $k = 1$. Since $\deg f \leq n$ the sum only goes up to $k = n$. So

$$f(x) = \sum_{k=1}^{n} \frac{f^{(k)}(1)}{k!} (x - 1)^k,$$

which proves that $\{ x - 1, (x - 1)^2, \ldots, (x - 1)^n \}$ spans $V$.

In Exercise 20.3 you showed that any set of polynomials $\{p_0, \ldots, p_n\}$ with $\deg p_k = k$ for all $k$ is linearly independent. Throwing out $p_0$, the set remains linearly independent. Since $\deg(x - 1)^k = k$, it follows that $\{ x - 1, (x - 1)^2, \ldots, (x - 1)^n \}$ is linearly independent and therefore is a basis of $V$, so $\dim V = n$. 
