Linear algebra Exam 2 Study Problems: Chapters 10, 13-18

1. Find a vector lying on the intersection of the two planes

\[ x + y + z = 0, \quad x + 2y + 3z = 0. \]

**Solution:** The line of intersection contains the cross-product of the normal vectors \( \mathbf{u} = (1, 1, 1) \) and \( \mathbf{v} = (1, 2, 3) \), so a desired vector is

\[ \mathbf{u} \times \mathbf{v} = (1, -2, 1). \]

Any nonzero scalar multiple of this is correct.

2. Find the cosine of the angle \( \theta \) between the vectors

\[ \mathbf{u} = (1, 1, 1), \quad \mathbf{v} = (1, 2, 3). \]

What is the area of the parallelogram spanned by \( \mathbf{u} \) and \( \mathbf{v} \)?

**Solution:**

\[ \cos \theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{|\mathbf{u}| |\mathbf{v}|} = \frac{6}{\sqrt{42}}. \]

The area is

\[ |\mathbf{u} \times \mathbf{v}| = |(1, -2, 1)| = \sqrt{6}. \]

3. Find the volume of the parallelopiped spanned by

\[ \mathbf{u} = (1, 1, 0), \quad \mathbf{v} = (1, 0, 1), \quad \mathbf{w} = (0, 1, 1). \]

**Solution:** The volume is

\[ \left| \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right| = 2. \]

4. Find the kernels of the following matrices.

\[
\begin{bmatrix}
4 & 5 & 2 \\
3 & 3 & 1 \\
1 & 2 & 1
\end{bmatrix},
\begin{bmatrix}
6 & -1 & -1 \\
2 & 7 & 3 \\
-1 & 2 & 1
\end{bmatrix},
\begin{bmatrix}
2 & 2 & 2 \\
1 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
Solution:

line: \( \mathbb{R}(1, -2, 3) \), line: \( \mathbb{R}(1, -5, 11) \), plane: \( x+y+z = 0 \), plane: \( y+z = 0 \).

5. Let \( A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{bmatrix} \). Find \( A^{-1} \) and the characteristic polynomial of \( A \).

Solution:

\[
\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ -3 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix}, \quad P_A(x) = x^3 - 10x^2 + 7x - 1
\]

6. Find the eigenvalues and eigenspaces of the matrices

\[
A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 0 & -11 \\ 0 & 1 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 10 & -10 \\ 10 & 22 & -20 \\ 15 & 30 & -28 \end{bmatrix}.
\]

Solution:

\( A \): Eval 1, 2, 3. \( E(1) = \mathbb{R}(6, -5, 1) \), \( E(2) = \mathbb{R}(3, -4, 1) \), \( E(3) = \mathbb{R}(2, -3, 1) \).

\( B \): Eval 2, 2, -3. \( E(2) = \text{plane } x + 2y - 2z = 0 \), \( E(-3) = \mathbb{R}(1, 2, 3) \).

7. Find a \( 3 \times 3 \) matrix whose eigenvalues are 1, -1, 2, with corresponding eigenvectors \((1, 2, 3), (1, 3, 5), (1, 3, 6)\).

Solution: Let

\[
B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{bmatrix}.
\]

Then \( B^{-1} \) was computed in problem 2. The desired matrix is

\[
A = B \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} B^{-1} = \begin{bmatrix} 8 & -8 & 3 \\ 21 & -23 & 9 \\ 36 & -42 & 17 \end{bmatrix}.
\]
8. Solve the system of equations

\begin{align*}
2x + y + z &= 5 \\
x + y + z &= 6 \\
x + y + 2z &= 7.
\end{align*}

Hint: To solve \(Ax = y\) for \(x\), compute \(A^{-1}\); the solution is \(x = A^{-1}y\).

**Solution:** The inverse of the coefficient matrix is

\[
\begin{bmatrix}
2 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 2
\end{bmatrix}^{-1} = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 3 & -1 \\
0 & -1 & 1
\end{bmatrix},
\]

and

\[
\begin{bmatrix}
1 & -1 & 0 \\
-1 & 3 & -1 \\
0 & -1 & 1
\end{bmatrix} \begin{bmatrix}
5 \\
6 \\
7
\end{bmatrix} = \begin{bmatrix}
-1 \\
6 \\
1
\end{bmatrix}
\]

so the solution is \(x = -1, \ y = 6, \ z = 1\).

9. We know that matrices

\[
A = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

are rotations, by angles of \(\pi/2\), and \(2\pi/3\) respectively. Find the axis and angle of rotation of \(AB\) and \(BA\).

**Solution:** \(AB\) is rotation by \(\pi\) about the vector \((0, 1, 1)\), while \(BA\) is rotation by \(\pi\) about the vector \((1, 0, 1)\).

10. Find the matrix that rotates about the axis \(\mathbb{R}(1, 0, 1)\) by an angle of \(\pi/4\) as \((1, 0, 1)\) points at you.

**Solution:** The unit vector is \(\frac{1}{\sqrt{2}}(1, 0, 1)\). Using the formula

\[A = I + \sin \theta U + (1 - \cos \theta)U^2,\]

where \(U\) is the unit vector and \(\theta\) is the angle of rotation, we can find the matrix for rotation about the axis.
we get
\[
A = I + \frac{1}{2} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \sqrt{2} - 1 \begin{bmatrix} \frac{-1}{2\sqrt{2}} & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}
\]
\[
= \frac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{2} + 1 & -\sqrt{2} & \sqrt{2} - 1 \\ \sqrt{2} & 2 & -\sqrt{2} \\ \sqrt{2} - 1 & \sqrt{2} & \sqrt{2} + 1 \end{bmatrix}
\]

11. Let \( \mathbf{u} \) be a unit vector in \( \mathbb{R}^3 \).

(a) Suppose \( \mathbf{w} \) in \( \mathbb{R}^3 \) is perpendicular to \( \mathbf{u} \). Show that
\[
\mathbf{u} \times (\mathbf{u} \times \mathbf{w}) = -\mathbf{w}.
\]

(b) Suppose \( \mathbf{v} \) is any vector in \( \mathbb{R}^3 \). Show that
\[
\mathbf{u} \times (\mathbf{u} \times (\mathbf{u} \times \mathbf{v})) = -\mathbf{u} \times \mathbf{v}.
\]

Solution: Part (a) can be done directly as follows. Let \( \mathbf{u} = (a, b, c) \) be a unit vector and let \( \mathbf{w} = (a', b', c') \). Since \( \mathbf{u} \) is a unit vector we have \( a^2 + b^2 + c^2 = 1 \).

And since \( \mathbf{w} \) is perpendicular to \( \mathbf{u} \), we have \( aa' + bb' + cc' = 0 \). Now compute:
\[
\mathbf{u} \times (\mathbf{u} \times \mathbf{w}) = (a, b, c) \times (bc' - cb', ca' - ac', ab' - ba')
\]
\[
= (b(ab' - ba') - c(ca' - ac'), c(bc' - cb') - a(ab' - ba'), a(ca' - ac') - b(bc' - cb'))
\]
\[
= (a(bb' + cc') - a'(b^2 + c^2), b(aa' + cc') - b'(a^2 + c^2), c(aa' + bb') - c'(a^2 + b^2))
\]
\[
= (a(-aa' - a'(1 - a^2), b(-bb') - b'(1 - b^2), c(-cc') - c'(1 - c^2))
\]
\[
= (-a', -b', -c') = -\mathbf{w}.
\]

Alternatively, note we could scale \( \mathbf{w} \) to be a unit vector. Then \( \mathbf{u}, \mathbf{w}, \mathbf{u} \times \mathbf{w} \) is a right-handed orthonormal basis, so following the cyclic order we have \( (\mathbf{u} \times \mathbf{w}) \times \mathbf{u} = \mathbf{w} \). This means \( \mathbf{u} \times (\mathbf{u} \times \mathbf{w}) = -\mathbf{w} \).

Part (b) follows immediately from (a) by taking \( \mathbf{w} = \mathbf{u} \times \mathbf{v} \).

12. Show that if \[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]
is a 2 \( \times \) 2 matrix with non-real eigenvalues then \( b \) and \( c \) have opposite signs.
Solution: To have complex eigenvalues we must have \( \text{tr}(A)^2 - 4 \det(A) < 0. \) Now

\[
\text{tr}(A)^2 - 4 \det(A) = (a+d)^2 - 4(ad-bc) = a^2+d^2-2ad+4bc = (a-d)^2 + 4bc.
\]

For this to be negative we need \( bc < 0, \) so \( b \) and \( c \) have opposite signs.

13. Is it possible for a \( 3 \times 3 \) real matrix to have all of its eigenvalues non-real?
Solution: No. The characteristic polynomial \( P_A(x) = x^3 + \ldots, \) so \( P_A(x) \to \pm \infty \) as \( x \to \pm \infty. \) By continuity, the graph of \( P_A(x) \) crosses the \( x \)-axis at some point. This point is a real eigenvalue for \( A. \) The other two eigenvalues are either both real or are complex conjugates.

14. Let \( A \) be a \( 3 \times 3 \) matrix and let \( u, v \) be vectors in \( \mathbb{R}^3. \)
(a) Show that \( \langle Au, v \rangle = \langle u, A^T v \rangle. \)
(b) Suppose \( A = A^T \) (we say \( A \) is “symmetric”). Show that two eigenvectors for \( A \) with distinct eigenvalues are perpendicular.
Solution: Since both sides of (a) are linear in \( u \) and \( v, \) it suffices to prove \( A \) for \( u = e_i \) and \( v = e_j \) for any \( i, j. \) On one hand, we have

\[
\langle Ae_j, e_i \rangle = \langle \text{col}_j(A), e_i \rangle = a_{ij},
\]

the entry in row \( i \) column \( j \) of \( A. \) On the other hand,

\[
\langle e_j, A^T e_i \rangle = \langle e_j, \text{col}_i(A^T) \rangle
\]

is the entry in row \( j \) column \( i \) of \( A^T, \) which is just \( a_{ij} \) again, by the definition of the transpose.

Now for (b). Suppose \( u \) and \( v \) are eigenvectors with distinct eigenvalues \( \lambda \) and \( \mu, \) respectively. Then

\[
\langle Au, v \rangle = \langle \lambda u, v \rangle = \lambda \langle u, v \rangle.
\]

But since \( A = A^T, \) we have

\[
\langle Au, v \rangle = \langle u, A^T v \rangle = \langle u, Av \rangle = \langle u, \mu v \rangle = \mu \langle u, v \rangle.
\]

Therefore

\[
\lambda \langle u, v \rangle = \mu \langle u, v \rangle.
\]

As \( \lambda \neq \mu, \) we must have \( \langle u, v \rangle = 0. \)