Chapter 2

SPECIAL TYPES OF MATRICES

A scalar matrix:

\[
\begin{bmatrix}
a & 0 \\
0 & a
\end{bmatrix}.
\]

A diagonal matrix:

\[
\begin{bmatrix}
a & 0 \\
0 & d
\end{bmatrix}.
\]

The product of two diagonal matrices is very simple:

\[
\begin{bmatrix}
a & 0 \\
0 & d
\end{bmatrix} \begin{bmatrix}
a' & 0 \\
0 & d'
\end{bmatrix} = \begin{bmatrix} aa' + 0 & 0 + 0 \\
0 + 0 & 0 + dd'
\end{bmatrix} = \begin{bmatrix} aa' & 0 \\
0 & dd'
\end{bmatrix}.
\]

Thus, diagonal matrices multiply like numbers. Note that if \( A \) and \( B \) are diagonal, then

\[ AB = BA. \]

An upper triangular matrix:

\[
\begin{bmatrix}
a & b \\
0 & d
\end{bmatrix}.
\]

In contrast to diagonal matrices, upper triangular matrices do not always commute with each other. For example:

\[
\begin{bmatrix}
2 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
2 & 2 \\
0 & 1
\end{bmatrix},
\]

while

\[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
2 & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
2 & 1 \\
0 & 1
\end{bmatrix}.
\]

A symmetric matrix:

\[ A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}. \]

You have, or will see these in multivariable calculus: The matrix of second partial derivatives of a function \( f(x, y) \) is

\[
\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.
\]
which is symmetric since $f_{xy} = f_{yx}$. Symmetric matrices will arise later in this course. The **transpose** of a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the matrix $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. So a matrix $A$ is symmetric exactly when $A = A^T$.

**A rotation matrix:**

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \text{ where } a^2 + b^2 = 1.$$  

Thus, $(a,b)$ is a point on the unit circle. Let $\theta$ be the angle of rotation from $(1,0)$ up to $(a,b)$. Then $a = \cos \theta, \ b = \sin \theta$, so

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$  

This $\theta$ is called the “angle of rotation”, for reasons that will become clear.

Examples of rotation matrices:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ here } \theta = \frac{\pi}{2}.$$  

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ here } \theta = \frac{\pi}{4}.$$  

**A reflection matrix:**

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \text{ where } a^2 + b^2 = 1.$$  

Note that $A^2 = I$.

In other words, reflection matrices are their own inverses.

Examples of reflection matrices:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$  

Rotations and reflections are exactly those matrices which satisfy $A^{-1} = A^T$ (inverse equals transpose). The difference between them is that

$$\det A = +1 \text{ if } A \text{ is a rotation,}$$  
$$\det A = -1 \text{ if } A \text{ is a reflection.}$$

**Exercise 2.1.** Show that scalar matrices commute with all other matrices.
Exercise 2.2. Suppose $a \neq d$. Show that the only matrices commuting with $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ are diagonal matrices.

Exercise 2.3. Show that $(AB)^T = (B^T)(A^T)$.

Note the reversal of order in the products.

Exercise 2.4.

a) Show that the product of two rotation matrices is again a rotation matrix.

b) Show that the product of two reflection matrices is a rotation matrix.

c) The product of a reflection matrix times a rotation matrix is what kind of matrix?

(Hint, use the fact, proved in the next chapter, that $\det(AB) = (\det A)(\det B)$.)

Exercise 2.5. Find a matrix $A$ such that $A^2 = I$, but which is not a reflection matrix.

Exercise 2.6. Find a rotation matrix whose entries are nonzero rational numbers.

(A rational number is a quotient of two integers. $\frac{1}{\sqrt{2}}$ is not a rational number.

Hint: Consider the 3-4-5 right triangle.)

Exercise 2.7. Suppose $A$ is a rotation or a reflection. Show that $A^T = A^{-1}$. 