Chapter 4

MATRICES AS LINEAR MAPS

A matrix can be used to move vectors\(^1\) around in the plane, as follows.

\[
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
\end{bmatrix}
= 
\begin{bmatrix}
ax + by \\
cx + dy \\
\end{bmatrix}.
\]

Thus, the matrix \(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\) moves the vector \((x, y)\) to the vector \((ax + by, cx + dy)\).

Different matrices move points around in different ways. For example, a rotation matrix

\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \\
\end{bmatrix}
\]

rotates every point by angle \(\theta\) in the counterclockwise direction.

The identity matrix \(I\) (which is a rotation matrix for \(\theta = 0\)) sends every point to itself. No point is moved by the identity matrix. The matrix \(-I\) is rotation by \(\theta = \pi\), so it sends every point to its antipode with respect to the origin.

On the other hand, will see in the next chapter that a reflection matrix moves vectors by reflecting them about a line through the origin. This is quite different from a rotation, because a reflection fixes (does not move) every vector on its reflecting line, while a rotation (except for \(I\)) fixes only the origin.

A matrix \(A\) is completely determined by what it does to the vectors \((1, 0)\) and \((0, 1)\). These two vectors are used so often that we give them permanent names:

\[
e_1 = (1, 0), \quad e_2 = (0, 1).
\]

Now, if \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\), then

\[
Ae_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}, \quad \text{and} \quad Ae_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.
\]

This simple observation is perhaps the

**Most Important Thing (about matrices).** \(Ae_1\) is the first column of \(A\) and \(Ae_2\) is the second column of \(A\).

\(^1\)In this course vectors are points, and we use the two words interchangeably. In other courses (especially Physics) a vector is an arrow with fixed length and direction, but which can lie anywhere in the plane. All of our vectors will have their base at the origin, hence are determined by their head, which is a point.
For example, what is the matrix that sends \((1, 0)\) to \((2, 3)\) and \((0, 1)\) to \((4, 1)\)? Answer: The matrix is
\[
\begin{bmatrix}
2 & 4 \\
3 & 1
\end{bmatrix}.
\]

What is the matrix that reflects about the line \(y = x\) and then reflects again about the line \(y = -x\)? Answer: If you draw a picture, you’ll see that \(e_1\) goes to \(-e_1\) and \(e_2\) goes to \(-e_2\). So the matrix is
\[
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix} = -I.
\]

**Exercise 4.1.**

(a) Find the matrix sends \((1, 0)\) to \((1, 2)\) and sends \((0, 1)\) to \((2, 1)\).

(b) Find the matrix that sends \((1, 0)\) to \((2, 1)\) and sends \((0, 1)\) to \((1, 1)\).

(c) Find the matrix that sends \((2, 1)\) to \((1, 2)\) and sends \((1, 1)\) to \((2, 1)\). Hint for (c): Use (a) and the inverse of the matrix in (b).

**Exercise 4.2.** A octagon has eight vertices, starting at \((1, 0)\) and rotating by multiples of \(\pi/4\). Compute the matrix \(A\) that does this rotation, and then compute \(A, A^2, \ldots, A^7\). Plot the first columns of these matrices. You should get to find the coordinates of the remaining seven vertices of the octagon.

**Exercise 4.3.** Find the matrix that rotates by \(\pi/4\) and then reflects about the line \(y = x\).

**Exercise 4.4.** Find the matrix that rotates by \(3\pi/4\) and then reflects about the line \(y = x\).

**Exercise 4.5.** If you take all the points on a line through the origin, and multiply them by a matrix, you will get another line. Let \(A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\). Describe how \(A\) moves the lines through the origin. (Hint: Start with the lines through \(e_1\) and \(e_2\), and then consider a line with slope \(m \neq 0, \infty\).)