FIBONACCI NUMBERS

The Fibonacci numbers are

\[ F_0 = 0, \quad F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \quad F_4 = 3, \quad F_5 = 5, \quad F_6 = 8, \ldots \]

To get the next Fibonacci number, you add the previous two. Thus, Fibonacci numbers are defined by the recursive formula

\[ F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_{n-1} + F_n. \]

To compute, say, \( F_{100} \) with this formula, you need to first compute \( F_{99} \) and \( F_{98} \), and for these you need to compute \( F_{97} \) and \( F_{96} \), and so on. Is there a formula that will tell us \( F_{100} \) without having to compute all the previous Fibonacci numbers? Yes. We’ll find it with matrices, using the same principle that we used to find the matrix of a reflection.

Let

\[ \Phi = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}. \]

( The Phibonacci matrix!) Then

\[ \Phi \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n-1} + F_n \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}. \]

Thus, applying \( \Phi \) to the vector whose components are a pair of successive Fibonacci numbers, we get the vector whose components are the next pair of successive Fibonacci numbers. For example,

\[ \Phi \begin{bmatrix} F_0 \\ F_1 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \]
\[ \Phi^2 \begin{bmatrix} F_0 \\ F_1 \end{bmatrix} = \Phi \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix}, \]
\[ \Phi^3 \begin{bmatrix} F_0 \\ F_1 \end{bmatrix} = \Phi \begin{bmatrix} F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} F_3 \\ F_4 \end{bmatrix}, \ldots \]

In general,

\[ \Phi^n \begin{bmatrix} F_0 \\ F_1 \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}. \]

But \( F_0 = 0 \) and \( F_1 = 1 \), so this just says that

\[ \Phi^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}. \]
This vector is the second column of $\Phi^n$. In fact,

$$
\Phi^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}.
$$

Therefore, if we want to compute $F_{100}$, “all” we have to do is compute $\Phi^{100}$.

As it stands, this is just a rephrasing of the problem, and we have not achieved much. If $\Phi$ were a diagonal matrix, then it would be easy to compute $\Phi^{100}$, but $\Phi$ is not a diagonal matrix. However, recall what happened with the reflection matrix $A$, in chapter 5. There was another matrix $B$ such that $B^{-1}AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, which is a diagonal matrix. So maybe there is a matrix $B$ such that $B^{-1}\Phi B$ is a diagonal matrix. Would that help us? Say that we find such a $B$, and that

$$
B^{-1}\Phi B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}.
$$

Then

$$
(B^{-1}\Phi B)^n = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}^n = \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix}.
$$

But notice also that

$$
(B^{-1}\Phi B)^n = (B^{-1}\Phi B)(B^{-1}\Phi B)(B^{-1}\Phi B) \cdots (B^{-1}\Phi B) = B^{-1}\Phi^n B. \tag{6b}
$$

Combining equations (6a) and (6b), we get

$$
B^{-1}\Phi^n B = \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix}.
$$

As we did with the reflection matrix, we extract $\Phi^n$ by multiplying on the left by $B$ and on the right by $B^{-1}$. This gives

$$
\Phi^n = B \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} B^{-1}. \tag{6c}
$$

And this is a formula for $\Phi^n$, which we have seen gives a formula or $F_n$. So we can solve the problem if we can find a matrix $B$ such that $B^{-1}\Phi B$ is diagonal:

$$
B^{-1}\Phi B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}.
$$

We first find $\lambda$ and $\mu$, using the fact that, no matter what $B$ is, we have

$$
\det(B^{-1}\Phi B) = \det(\Phi), \quad \text{and} \quad \tr(B^{-1}\Phi B) = \tr(\Phi).
$$

Since

$$
\det(B^{-1}\Phi B) = \lambda\mu, \quad \text{and} \quad \det(\Phi) = -1,
$$

and

$$
\tr(B^{-1}\Phi B) = \lambda + \mu, \quad \text{and} \quad \tr(\Phi) = 1,
$$

so

$$
\lambda\mu = -1, \quad \text{and} \quad \lambda + \mu = 1.
$$

Solving this system of equations, we find that $\lambda = \phi$ and $\mu = \phi^{-1}$, where $\phi = \frac{1 + \sqrt{5}}{2}$ is the golden ratio. Therefore, the matrix $B$ is

$$
B = \begin{bmatrix} \phi & 1 \\ 1 & \phi^{-1} \end{bmatrix}.
$$

Since $B^{-1}\Phi B$ is diagonal, we can compute $\Phi^n$ by simply multiplying $B^{-1}$ on the right and $B$ on the left:

$$
\Phi^n = B \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} B^{-1} = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}.
$$

Since $\Phi^n = F_n$, we have

$$
F_n = \begin{bmatrix} \phi^n & \phi^{n-1} \\ \phi^{n-1} & \phi^n \end{bmatrix}.
$$

This is the desired formula for $F_n$. The idea here is that, by finding a suitable matrix $B$, we can diagonalize $\Phi$, and then compute $\Phi^n$ easily by raising the diagonal matrix to the $n$th power. Then, by undoing the diagonalization, we obtain the formula for $F_n$. The details of this process are filled in the next section.
the unknowns $\lambda$ and $\mu$ satisfy the equations

$$\lambda \mu = -1, \quad \text{and} \quad \lambda + \mu = 1.$$ 

In other words, $\lambda$ and $\mu$ are the roots of the polynomial

$$P_\Phi(x) = (x - \lambda)(x - \mu) = x^2 - (\lambda + \mu)x + \lambda \mu = x^2 - x - 1.$$ 

Applying the quadratic formula, the roots of $P_\Phi(x)$ are

$$\frac{1}{2}(1 \pm \sqrt{5}).$$ 

We take one root to be $\lambda$, and the other to be $\mu$, say

$$\lambda = \frac{1}{2}(1 + \sqrt{5}), \quad \text{and} \quad \mu = \frac{1}{2}(1 - \sqrt{5}).$$ 

These numbers $\lambda$ and $\mu$ are called the eigenvalues of $\Phi$. Because they are the roots of $x^2 - x - 1$, we have

$$\lambda^2 = \lambda + 1, \quad \mu^2 = \mu + 1.$$ 

Now define vectors

$$u = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ \mu \end{bmatrix}.$$ 

Note that

$$\Phi u = \begin{bmatrix} \lambda \\ 1 + \lambda \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \lambda \end{bmatrix} = \lambda u,$$

so

$$\Phi u = \lambda u, \quad (6d)$$

and likewise

$$\Phi v = \mu v. \quad (6e)$$

The vectors $u$ and $v$ are called eigenvectors of $\Phi$.

Let $B$ be the matrix whose first column is $u$ and whose second column is $v$. In other words,

$$B = \begin{bmatrix} 1 & 1 \\ \lambda & \mu \end{bmatrix}.$$ 

As always, this means that $Be_1 = u$ and $Be_2 = v$.

We must check that $B^{-1}\Phi B$ is diagonal. We could compute $B^{-1}$ and just multiply all three matrices, but it is easier to proceed as we did with the reflection matrix, and calculate the effect of $B^{-1}\Phi B$ on the standard basis vectors $e_1, e_2$. Using equation (6d), we get

$$B^{-1}\Phi Be_1 = B^{-1}\Phi u = B^{-1}(\lambda u) = \lambda B^{-1}u = \lambda e_1,$$

and likewise, using (6e), we get

$$B^{-1}\Phi Be_1 = \mu e_2.$$
Thus,

\[ B^{-1} \Phi B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}. \]

Rejoice, we conquer! Now, remember, we wanted to compute \( \Phi^n \). Equation (6c) says that

\[ \Phi^n = B \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} B^{-1} = \begin{bmatrix} 1 & 1 \\ \lambda & \mu \end{bmatrix} \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \lambda & \mu \end{bmatrix}^{-1}. \]

Now \( \det \begin{bmatrix} 1 & 1 \\ \lambda & \mu \end{bmatrix} = -1/\sqrt{5} \), so

\[ \begin{bmatrix} 1 & 1 \\ \lambda & \mu \end{bmatrix}^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} -\mu & 1 \\ \lambda & -1 \end{bmatrix}, \]

so we get, finally,

\[ \Phi^n = \begin{bmatrix} 1 & 1 \\ \lambda & \mu \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -\mu & 1 \\ \lambda & -1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \mu^n \lambda - \lambda^n \mu & \lambda^n - \mu^n \\ \mu^{n+1} \lambda - \lambda^{n+1} \mu & \lambda^{n+1} - \mu^{n+1} \end{bmatrix}. \]

Since \( \lambda \mu = -1 \), we can write this more simply as

\[ \Phi^n = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda^{n-1} - \mu^{n-1} & \lambda^n - \mu^n \\ \lambda^n - \mu^n & \lambda^{n+1} - \mu^{n+1} \end{bmatrix}. \]

Recall that the upper right entry of \( \Phi^n \) is \( F_n \), so we get

\[ F_n = \frac{1}{\sqrt{5}} (\lambda^n - \mu^n). \quad \text{(Fibonacci Formula)} \]

Recall that

\[ \lambda = \frac{1}{2} (1 + \sqrt{5}) = 1.618 \ldots. \]

and

\[ \mu = \frac{1}{2} (1 - \sqrt{5}) = -0.618 \ldots. \]

If \( n \) is large in the Fibonacci Formula (which is the case we are interested in), then \( \mu^n \) will be small, so \( F_n \) will be close to \( \frac{1}{\sqrt{5}} (\lambda^n) \).

This gives a practical method for computing \( F_n \) for large \( n \): First compute \( \frac{1}{\sqrt{5}} (\lambda^n) \), then take the nearest integer.

For example, to compute \( F_{100} \) without computing any other Fibonacci numbers: We first compute

\[ \frac{\lambda^{100}}{\sqrt{5}} = 3.5422484817926191507500000 \times 10^{20}. \]

We need enough decimal places in order to tell what the nearest integer is; in this case we need at least 20 decimal places (to be safe, I took 25). So

\[ F_{100} = 3.54224848179261915075 \times 10^{20} = 354224848179261915075. \]
Exercise 6.1.

(a) Use this method to compute $F_{20}$, $F_{21}$ and $F_{30}$ without computing any other Fibonacci numbers.

(b) Compute $F_{30}$ again by starting with your values for $F_{20}$ and $F_{21}$ from part (a) and then computing $F_{22}$, $F_{23}$, ..., $F_{30}$ using the recursive formula $F_{n+1} = F_{n-1} + F_n$.

Exercise 6.2. Show that

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \lambda.$$

Exercise 6.3. The eigenvectors $u, v$ of $\Phi$ live on the lines with equations $y = \lambda x$, $y = \mu x$, respectively. Consider the function

$$f(x, y) = (y - \lambda x)(y - \mu x)$$

which is zero on these lines.

(a) Show that $f(x, y) = y^2 - xy - x^2$.

(b) Given a point $(x, y)$, define $(x', y')$ by the equation

$$\Phi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Show that

$$f(x', y') = -f(x, y).$$

(c) Let $C_+$ be the graph of the equation $y^2 - xy - x^2 = 1$, and let $C_-$ be the graph of the equation $y^2 - xy - x^2 = -1$. These graphs are hyperbolas, having the lines $y = \lambda x$, $y = \mu x$ as asymptotes. Use part (b) to show that the point $(F_n, F_{n+1})$ lives on $C_+$ if $n$ is even and on $C_-$ if $n$ is odd.

A picture to accompany this exercise will be drawn in class.

Exercise 6.4. The “Lucas numbers” are defined like the Fibonacci, except the two starting values are $L_0 = 2$, $L_1 = 1$. Thus, $L_2 = 3$, $L_3 = 4$, $L_4 = 7$, etc. Using the method of this chapter, find a formula for $L_n$ that does not require computing any other Lucas numbers. (Note the same matrix $\Phi$ is used, but the initial vector is different).

Exercise 6.5. Consider the sequence of numbers $a_n$ defined by

$$a_0 = 0, \quad a_1 = 1, \quad a_{n+1} = a_{n-1} + 2a_n.$$

Find a formula for $a_n$ that does not require computing earlier terms in the sequence, and compute

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n}.$$