Exercise 8.1. *Find the eigenvalues and eigenvectors of the following matrices.*

a) 

\[
A = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

*Solution:* \( \lambda = 1, \mu = -1, \mathbf{u} = (1, 1), \mathbf{v} = (1, -1). \)

b) 

\[
A = \begin{bmatrix}
3 & 4 \\
2 & -3
\end{bmatrix}
\]

*Solution:* \( \lambda = \sqrt{17}, \mu = -\sqrt{17}, \mathbf{u} = (4, \sqrt{17} - 3), \mathbf{v} = (4, -\sqrt{17} - 3). \)

c) 

\[
A = \begin{bmatrix}
3 & 0 \\
2 & -3
\end{bmatrix}
\]

*Solution:* \( \lambda = 3, \mu = -3, \mathbf{u} = (3, 1), \mathbf{v} = (0, 1). \)

d) 

\[
A = \begin{bmatrix}
2 & 1 \\
1 & 1
\end{bmatrix}
\]

*Solution:* \( \lambda = \frac{1}{2} [3 + \sqrt{5}], \mu = \frac{1}{2} [3 - \sqrt{5}], \mathbf{u} = (1, \frac{1}{2} [1 - \sqrt{5}]), \mathbf{v} = (1, \frac{1}{2} [1 + \sqrt{5}]). \)

e) 

\[
A = \begin{bmatrix}
15 & -10 \\
21 & -14
\end{bmatrix}
\]

*Solution:* \( \lambda = 1, \mu = 0, \mathbf{u} = (5, 7), \mathbf{v} = (2, 3). \)

Exercise 8.2. *Suppose A is a reflection matrix*  

\[
A = \begin{bmatrix}
a & b \\
b & -a
\end{bmatrix}, \quad a^2 + b^2 = 1.
\]

a) *Find the eigenvalues and eigenvectors of A.*

b) *Find a vector on the reflecting line of A.*

c) *Find a vector perpendicular to the reflecting line of A.*

All of your answers will involve \( a \) and \( b \).

*Solution:*

\( \lambda = 1 \) with eigenvector \( \mathbf{u} = (b, 1 - a). \)

\( \mu = -1 \), with eigenvector \( \mathbf{v} = (b, -1 - a). \)

\( \mathbf{u} = (b, 1 - a) \) is on the reflecting line, and

\( \mathbf{v} = (b, -1 - a) \) is perpendicular to the reflecting line.
Exercise 8.3. Suppose $A$ is a matrix whose entries you do not know, but you do know that its eigenvalues are $\lambda = 2$ and $\mu = 3$, with corresponding eigenvectors $u = (1, 3)$, $v = (6, -1)$.

a) Find $A$.

Solution:

$$A = \frac{1}{19} \begin{bmatrix} 56 & -6 \\ -3 & 39 \end{bmatrix}.$$  

b) Find the eigenvalues and eigenvectors of $A^{100}$.

Solution: $\lambda^{100}, \mu^{100}$, same eigenvectors as $A$.

Exercise 8.4. Let

$$A = \begin{bmatrix} 16 & -10 \\ 21 & -13 \end{bmatrix}.$$  

Find an explicit formula for $A^n$.

Solution:

$$A^n = \begin{bmatrix} 15 \cdot 2^n - 14 & 10(1 - 2^n) \\ 21(2^n - 1) & 15 - 14 \cdot 2^n \end{bmatrix}.$$  

Exercise 8.5. Suppose $A$ is a matrix with distinct eigenvalues $\lambda$ and $\mu$. Show that $\text{tr}(A) = \lambda + \mu$ and $\det(A) = \lambda \mu$.

Solution: Look at the characteristic polynomial in two ways. On the one hand, by definition we have

$$P_A(x) = x^2 - \text{tr}(A)x + \det(A).$$

On the other hand the roots of $P_A(x)$ are $\lambda$ and $\mu$, so

$$P_A(x) = (x - \lambda)(x - \mu) = x^2 - (\lambda + \mu)x + \lambda \mu.$$  

Comparing coefficients, we find

$$\text{tr}(A) = \lambda + \mu, \quad \det(A) = \lambda \mu,$$

as desired.