Exercise 9.1. For each of the following matrices $A$, find the eigenvalue $\lambda$, and a matrix $B$ such that $B^{-1}AB = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

(a) $A = \begin{bmatrix} 6 & 1 \\ -1 & 8 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

(To make life easier for the grader, please choose $v = e_1$ in each problem. Your $B$’s will all be the same, having 1 in the upper right corner.)

Solutions:

a) $7$, $B = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

b) $0$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

c) $1$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

d) $1$, $B = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

In general, $B = \begin{bmatrix} (a-d)/2 \\ c \end{bmatrix}$, if $A$ is not already upper triangular.

Exercise 9.2. Show that every eigenvector of $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ is proportional to $e_1$. (Hint: Just calculate.)

Solution: Set $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$.

Get $\begin{bmatrix} \lambda x + y \\ \lambda y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$,

so $y = 0$, and $(x, 0) = xe_1$.

Exercise 9.3. Let $A$ be a matrix with multiple eigenvalue $\lambda$, and let $u$ be an eigenvector of $A$. Assume that $A$ is not a scalar matrix. Prove the following statements.

(a) Every eigenvector of $A$ is proportional to $u$. (Hint: Proposition 1.)

(b) $A_0u = (0, 0)$

(c) If $v$ is a vector for which $A_0v = (0, 0)$, then $v$ is an eigenvector of $A$. 1
(Taken together, these facts mean that $A$ has exactly one eigensystem, which consists precisely of the vectors sent to $(0,0)$ by $A_0$.)

**Solutions:**

a) If $v$ is any eigenvector of $A$, then $B^{-1}v$ is an eigenvector of $B^{-1}AB$. But by exercise 9.2, all eigenvectors of $B^{-1}AB$ are proportional to $e_1$. So $B^{-1}v = te_1$ for some scalar $t$. Multiplying both sides by $B$, we get

$$v = tBe_1 = tu,$$

so $v$ is proportional to $u$.

b) Apply both sides of the matrix equation $A = \lambda I + A_0$ to $u$. We get

$$Au = \lambda u + A_0u.$$  

But $Au = \lambda u$, so

$$\lambda u = \lambda u + A_0u.$$  

Subtract $\lambda u$ from both sides, and get $A_0u = 0$.

c) If $A_0v = (0,0)$, then $Av = (\lambda I + A_0)v = \lambda v + (0,0) = \lambda v$.

**Exercise 9.4.** Suppose $A^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for some $n \geq 0$.

(a) What are the eigenvalues of $A$? (Hint: Apply $A$ repeatedly to both sides of the equation $Au = \lambda u$.)

(b) Compute $A^2$. (Hint: Use Proposition 3, and the value of $\lambda$ from part (a).)

**Solutions:**

a) $\lambda^n = 0$, so $\lambda = 0$.

b) $A = A_0$, so $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, by Prop. 3.

**Exercise 9.5.** Suppose $\lambda = 0$ is the only eigenvalue of $A$. Does this imply that $A$ is nilpotent?

**Solution:** Yes, because $A = A_0$, and $A_0^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

**Exercise 9.6.** Show that the nilpotent matrices are precisely those of the form

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$  

with $a^2 + bc = 0$.

(You have to show two things: First, that every nilpotent matrix has this form. Second, that every matrix of this form is nilpotent.)

**Solution:** First suppose $A$ is nilpotent. Then $\lambda = 0$ is the only eigenvalue of $A$, by exercise 9.4a). Hence $P_A(x) = x^2$, so $\text{tr} A = \det A = 0$. The former means $d = -a$, the latter means $-a^2 - bc = 0$.

Second, suppose

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$  

with $a^2 + bc = 0$.

then $P_A(x) = x^2$ so $\lambda = 0$ is the only eigenvalue, so $A$ is nilpotent, by the affirmative answer to exercise 9.5.