Linear Algebra
Study problems for Exam 1

1. a) Suppose a matrix $A$ has eigenvalues 2 and 3. What is the characteristic polynomial of $A^{-1}$?

b) Suppose $A^2 = I$. What are the possible characteristic polynomials of $A$? (There are three possibilities.)

c) Suppose $A$ has eigenvalues 2 and 3. What are the eigenvalues of $A^T$?

2. Find the matrix of the linear map that sends $(1, 1)$ to $(-1, -1)$ and $(1, 0)$ to $(1, 0)$. You can check your answer, and make up more problems of this kind.

3. Suppose that $A$ is a rotation matrix and $B$ is a reflection matrix.
   a) Show that $BAB^{-1} = A^{-1}$. (Hint: What kind of matrix is $BA$?)
   b) Show that $ABA^{-1}$ is a reflection matrix. What is the relation between the reflecting lines of $B$ and $ABA^{-1}$?

4. Find the matrix reflecting about the line $y = 3x$. Your answer should have rational entries.

5. a) Suppose $A$ has $\lambda = 0$ as an eigenvalue. What is det($A$)?

b) Suppose both eigenvalues of $A$ are zero. What is tr($A$)?

c) Find a matrix with all entries nonzero, both of whose eigenvalues are 0.

6. Express the vector $w = (3, 7)$ as a linear combination of $u = (1, 2)$ and $v = (1, 3)$. (Make up different vectors and do more of these.)

7. Two lines $\ell$ and $\ell'$ through the origin make an angle of $\pi/4$, going clockwise from $\ell$ to $\ell'$. Let $A$ and $A'$ be the corresponding reflection matrices. Find the matrix $AA'$.

8. Consider the following sequence of numbers:

   $x_0 = 0, x_1 = 1, x_2 = 1, x_3 = 3, x_4 = 5, x_5 = 11, x_6 = 21, \ldots$

   Find a formula for $x_n$ that does not involve $x_k$ for $k < n$.

9. Suppose $P = 400$ students. Each month 80% of those in Dorms move to Apts and 20% of those in Apts move to Dorms.
   a) Find the migration matrix and its eigenvalues and eigenvectors.
   
   b) If the initial distribution is $(200, 200)$, find a formula for the population distribution after $n$-months. (Do it from scratch, without relying on the formulas from chapter 7.)
   
   c) What is the stable equilibrium point? What initial distribution will take longest to reach the stable equilibrium point?
10. For each of the following matrices $A$, find a matrix $B$ such that $B^{-1}AB$ is diagonal.

\[
\begin{align*}
A &= \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, & A &= \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, & A &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, & A &= \begin{bmatrix} 0 & b \\ b^{-1} & 0 \end{bmatrix} \quad (b \neq 0).
\end{align*}
\]

11. Make up two nonproportional vectors $\mathbf{u}$ and $\mathbf{v}$, and distinct numbers $\lambda$ and $\mu$.

a) Find a matrix $A$ such that $A\mathbf{u} = \lambda\mathbf{u}$ and $A\mathbf{v} = \lambda\mathbf{v}$.

b) Find other matrices with the same eigenlines, but different eigenvalues.

c) Find other matrices with the same eigenvalues, but different eigenlines.

d) Can you find other matrices with the same eigenlines and the same eigenvalues?