Math 202 extra problems for Note 6

1. Let $R$ be the cube $0 \leq x, y, z \leq 1$, let $S$ be the boundary of $R$, and let $\mathbf{F} = (2x - z, x^2y, -xz^2)$.
   a) Compute $\iint_S \mathbf{F} \cdot \mathbf{N} \, dS$
   b) Compute $\iiint_R \nabla \cdot \mathbf{F} \, dR$.

   According to the Divergence Theorem, you should get the same answer for both, namely $\frac{11}{6}$.

2. Let $R$ be the part of the ball $x^2 + y^2 + z^2 \leq 1$ with $x, y, z \geq 0$, and let $S$ be the boundary of $R$.
   a) Calculate $\iiint_R x^{n-1}y^{m-1}z^{k-1} \, dR$.
   b) Let $\mathbf{F} = \frac{1}{n}(x^ny^mz^k, 0, 0)$. Calculate $\iint_S \mathbf{F} \cdot \mathbf{N} \, dS$.

   According to the Divergence Theorem, you should get the same answer for both, namely

   \[
   \frac{(\frac{n}{2} - 1)!(\frac{m}{2} - 1)!(\frac{k}{2} - 1)!}{8(\frac{n+m+k}{2})!}.
   \]

3. Let $B^2$ be the unit disk in $\mathbb{R}^2$. Compute the average radius of a point in $B^2$. Do the same for the unit ball $B^3$ in $\mathbb{R}^3$. Now using equation (4e) in Note 4, compute the average radius of a point in $B^n$. You should find that the average radius goes to 1 as $n$ gets large. In other words, more and more of the ball is concentrated near the boundary.

4. Let $R$ be the top half of the ball of radius $a$, given by $x^2 + y^2 + z^2 \leq a^2$, $z \geq 0$. Use the divergence theorem to compute

   \[
   \iint_S xz^2dydz + (x^2y - z^3)dzdx + (2xy + y^2z)dxdy.
   \]

   (You can check your answer by computing the surface integral directly.)

5. Let $\mathbf{c}(t) = (x(t), 0, z(t))$, $a \leq t \leq b$ be a curve in the $xz$ plane, with $x(t)$ always positive. Revolve $\mathbf{c}$ around the $z$ axis to get a surface $S$. Show that the area of $S$ is the length of $\mathbf{c}$ times the circumference of the circle travelled around the $z$ axis by the center of mass of $\mathbf{c}$.

   To get started, here is a parametrization of $S$:

   \[
   \mathbf{r}(t, \theta) = (x(t) \cos \theta, x(t) \sin \theta, z(t)), \quad 0 \leq \theta \leq 2\pi, \quad a \leq t \leq b.
   \]

   Check that the result is true for the sphere $S_a$ (obtained by revolving a semicircle) and the torus $T_{ab}$ (obtained by revolving a circle). This was discovered by an ancient Greek mathematician named Pappus, before the invention of surface integrals.

6. Let $S$ be the triangle in space with vertices at $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Choose the upward pointing normal vector. Let $C$ be the boundary of $S$. Let $\mathbf{F} = (z, x, y)$.
   a) Compute $\mathbf{\nabla} \times \mathbf{F}$.
   b) Compute $\iint_S \mathbf{\nabla} \times \mathbf{F} \cdot N \, dS$. 

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c) Compute $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$.

According to Stokes theorem, you should get the same answers for a) and b).

Stokes theorem implies that if $\nabla \times \mathbf{F} = 0$, then $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$ depends only on the endpoints of $C$. Use this fact in the next two problems.

7. Let $\mathbf{F} = (2xz^3 + 6y, 6x - 2yz, 3x^2z^2 - y^2)$.
   a) Compute $\nabla \times \mathbf{F}$.
   b) Use the result of a) to compute $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$ where $C$ is a squiggly path from $(1, -1, 1)$ to $(2, -1, 1)$. (Ans: 15).
   c) Can you do b) with $\mathbf{F}$ replaced by $\mathbf{G} = (0, 0, y)$?

8. Let $r = (x^2 + y^2 + z^2)^{1/2}$. Let $p$ be a point on $S_a$, and let $q$ be a point on $S_b$, where $a < b$, and let $C$ be any path from $p$ to $q$. Compute

$$\int_C x r \, dx + x r \, dy + x r \, dz.$$  

(Easiest method: Find a potential function for $\mathbf{F} = (x r, y r, z r)$. Answer: $(b^3 - a^3)/3$.)